Asymmetry

Picture by M.Miliaresi's graduate thesis "Dynamics of asteroids in resonance with Jupiter".

Asymmetry is an electronic mathematical journal which aims at giving the chance to the readers, and the editors themselves, to work on interesting mathematical problems or find information about various mathematical topics. The problems presented may either be original or taken from the existing literature or the web. Attempt will be made towards the precision of the problems' original source. The topics' level is undergraduate and beyond. Readers are encouraged by the editors to submit proposals and/or solutions to proposed problems. Proposals and solutions are preferred to be in LaTeX format using what is necessary from the preamble presented in http://akotronismaths.blogspot.gr/p/asymmetry-electronic-mathematical.html, must be legible and should appear on separate sheets, each indicating the name of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that no solution is available at the time the problem is published. Questions concerning proposals and/or solutions can be sent by e-mail to akotronis@gmail.com.

Problems and Solutions
The source of the problems will appear along with the publication of the solutions

The Problems

V2-1 Let $f : [-1, 1] \to \mathbb{R}$ be an odd and Riemann integrable function such that $\int_0^{2k\pi} x^2 f(\sin x) \, dx \neq 0$ for $k \in \mathbb{N}$. Evaluate

$$\sum_{k \geq 1} \frac{\int_0^\pi f(\sin x) \, dx}{\int_0^{2k\pi} x^2 f(\sin x) \, dx}.$$

V2-2 *Proposed by Spyros Kapellides Ioannina Greece*

Let $p(x)$ be a polynomial with real coefficients such that $\{p(n)\} < \dfrac{1}{n}$, $\forall n \in \mathbb{N}$. Show that $p(n) \in \mathbb{Z}$, $\forall n \in \mathbb{N}$.

$\{\cdot\}$ denotes the fractional part.

V2-3 Let x_n the sequence defined by $x_n = x_{n-1}^2 - 2$, $n \geq 1$ and $x_0 = 3$. Evaluate

$$\sum_{n \geq 0} \left(\prod_{k=0}^n x_k \right)^{-1},$$

if the series converges.

V2-4 Let $a_n = \left(\prod_{k=0}^{n} \binom{n}{k} \right)^{\frac{1}{n(n+1)}}$.

  1. Show that $\lim_{n \to +\infty} a_n = 1$ and

  2. evaluate $\lim_{n \to +\infty} \dfrac{n(a_n - 1)}{\ln n}$, if it exists.

V2-5 Evaluate $\displaystyle\int_0^1 \sqrt{4x - 4x^2} \tanh^{-1}\left(\sqrt{4x - 4x^2} \right) dx$.

V2-6 Let k be a positive integer. Show that

$$\sum_{n \geq 1} \frac{1}{(2n-1)(2n-3)\cdots(2n-2k-1)} = \frac{(-1)^k 2^{k-1}}{k \cdot k! \binom{2k}{k}}.$$

V2-7 Evaluate $\displaystyle\sum_{k=0}^{n} (-1)^{k+1} \frac{\binom{n}{k}}{2k-1}$.

V2-8 Let $A_{n,m,k} := \dfrac{(-1)^{m-1}}{m 2^n n^m} \binom{n}{k} k^m$, where m, n are positive integers and k is a non-negative integer.

  1. Can we find a sequence $\{a_m\}_{m \geq 1}$ and $n_0 \in \mathbb{N}$ such that $\mathbb{N} \ni n \geq n_0 \Rightarrow \left| \sum_{k=0}^{n} A_{n,m,k} \right| < a_m$ for

     every m, with $\displaystyle\sum_{m=1}^{+\infty} a_m$ being convergent?

  2. (*) Is it true that, in the case that $\lim_{n \to +\infty} \sum_{k=0}^{n} A_{n,m,k} = a_m \in \mathbb{R}$ $m \geq 1$ with $\displaystyle\sum_{m \geq 1} a_m$ convergent,

     then $\lim_{n \to +\infty} \sum_{k=0}^{n} \sum_{m \geq 1} A_{n,m,k} = \sum_{m \geq 1} a_m$?

  3. Evaluate

$$\lim_{n \to +\infty} \frac{1}{n} \left(\prod_{\ell=0}^{n} (n+\ell)^{C_n^\ell} \right)^{\frac{1}{2n}}, \qquad \text{where } C_n^\ell = \binom{n}{\ell},$$

     if it exists.

Solutions

V1-1 Evaluate

$$\sum_{n \geq 0} \sum_{k=0}^{n} \frac{(-4)^k}{(2k+1)} \frac{\binom{n}{k}}{\binom{2k}{k}} x^n$$

for the real values of x that the sum converges.

Solution: Omran Kouba, *Higher Institute for Applied Sciences and Technology, Damascus, Syria*

$$\int_0^1 x^k(1-x)^k\,dx = \beta(k+1,k+1) = \frac{\Gamma(k+1)\Gamma(k+1)}{\Gamma(2k+2)} = \frac{(k!)^2}{(2k+1)!} = \frac{1}{(2k+1)\binom{2k}{k}}$$

It follows that

$$\sum_{k=0}^n \frac{(-4)^k}{(2k+1)} \frac{\binom{n}{k}}{\binom{2k}{k}} = \sum_{k=0}^n \binom{n}{k}(-4)^k \int_0^1 x^k(1-x)^k\,dx$$

$$= \int_0^1 \left(\sum_{k=0}^n \binom{n}{k}(-4x(1-x))^k \right) dx = \int_0^1 (1-4x+4x^2)^n\,dx$$

$$= \int_0^1 (2x-1)^{2n}dx = \left[\frac{(2x-1)^{2n+1}}{2(2n+1)} \right]_{x=0}^{x=1} = \frac{1}{2n+1}$$

Thus, the considered sum converges only for $x \in [-1,1[$, and we have

$$\sum_{n=0}^\infty \sum_{k=0}^n \frac{(-4)^k}{(2k+1)} \frac{\binom{n}{k}}{\binom{2k}{k}} x^n = \sum_{n=0}^\infty \frac{x^n}{2n+1}$$

Recalling that

$$\sum_{n=0}^\infty \frac{t^{2n+1}}{2n+1} = \tanh^{-1}(t) \quad \text{and} \quad \sum_{n=0}^\infty \frac{(-1)^n t^{2n+1}}{2n+1} = \arctan(t)$$

we conclude that, for $x \in [-1,1[$, we have

$$\sum_{n=0}^\infty \sum_{k=0}^n \frac{(-4)^k}{(2k+1)} \frac{\binom{n}{k}}{\binom{2k}{k}} x^n = \begin{cases} \dfrac{\tanh^{-1}\sqrt{x}}{\sqrt{x}} & \text{if} \quad x \in (0,1), \\ 0 & \text{if} \quad x=0, \\ \dfrac{\arctan\sqrt{-x}}{\sqrt{-x}} & \text{if} \quad x \in [-1,0), \end{cases}$$

which is the desired conclusion. $\square$

V1-2 Evaluate

$$\sum_{k\geq 0} \binom{n+k}{2k}\binom{2k}{k}\frac{(-1)^k}{k+1}$$

for $n \in \mathbb{Z}$.

Solution 1: Omran Kouba, *Higher Institute for Applied Sciences and Technology, Damascus, Syria*

The basic ingredient is the following Lemma:

Lemma : If P is a polynomial such that $\deg P < n$ then

$$\sum_{k=0}^n (-1)^k \binom{n}{k} P(k) = 0$$

Proof. Indeed, using linearity, we only have to show that

$$\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}k^{p}=0,\quad\text{for }p\in\{0,1,\ldots,n-1\}.\tag{1}$$

Consider $F(z)=(1-e^{z})^{n}$. Then F is an entire function having 0 as a zero of order n. This implies that $F^{(p)}(0)=0$ for $p\in\{0,1,\ldots,n-1\}$, and (1) follows since $F(z)=\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}e^{kz}$. $\qquad\square$

Let us denote the considered sum by S_{n}, it is a finite sum, since the nonzero terms correspond only to the values of k that belong to $\{0,1,\ldots,\max(n,-n-1)\}$. Recall that for a nonnegative integer p we have

$$\binom{X}{0}=1,\text{ and for }p>0,\quad\binom{X}{p}=\frac{X(X-1)\cdots(X-p+1)}{p!}$$

In particular, for $k>0$, we have

$$\binom{-n+k}{2k}=\frac{(-n+k)(-n+k-1)\cdots(-n-k+1)}{(2k)!}$$
$$=\frac{(n+k-1)(n+k-2)\cdots(n-k)}{(2k)!}=\binom{n-1+k}{2k}$$

This proves that $S_{-n}=S_{n-1}$. And since clearly we have $S_{0}=1$, it is sufficient to consider the case $n>0$. But, for $n\geq k\geq 0$, $n\geq 1$ we have

$$\binom{n+k}{2k}\binom{2k}{k}\frac{1}{k+1}=\frac{(n+k)!}{(k+1)!n!}\binom{n}{k}=P_{n}(k)\binom{n}{k}$$

Where

$$P_{n}(X)=\frac{1}{n!}\prod_{2\leq j\leq n}(X+j).^{1}$$

Thus, $S_{n}=\sum_{k=0}^{n}(-1)^{k}P_{n}(k)\binom{n}{k}$. Now, the fact that $\deg P_{n}\leq n-1$ implies that $S_{n}=0$. This follows from the Lemma. Finally, $S_{n}=1$ for $n\in\{-1,0\}$, and $S_{n}=0$ for $n\notin\{-1,0\}$. $\qquad\square$

Solution 2: A.KOTRONIS

We use that

$$\sum_{k\geq 0}(-1)^{k}\binom{2k}{k}\frac{z^{k}}{k+1}=\frac{\sqrt{1+4z}-1}{2z},\qquad|z|<1/4\tag{I1}$$

which easily deduced by $\sum_{k\geq 0}\binom{2k}{k}z^{k}=(1+4z)^{-1/2},\qquad|z|<1/4.$

As shown in the first solution, and keeping the same notation for S_{n}, it is $S_{-n}=S_{n-1}$ and trivially $S_{0}=1$.

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[1]$P_{n}(X)$ is considered to be 1 when $n=1$.

Now for $n \geq 1$ denoting by C a circle centered at the origin with radius $r > 2(\sqrt{2}+1)$, we have

$$S_n = \frac{1}{2\pi i} \sum_{k \geq 0} \int_C \frac{(z+1)^{n+k}}{z^{2k+1}} \, dz \stackrel{2}{=} \frac{1}{2\pi i} \int_C \frac{(z+1)^n}{z} \sum_{k \geq 0} \binom{2k}{k} \frac{(-1)^k}{k+1} \left(\frac{z+1}{z^2}\right)^k dz$$

$$\stackrel{\text{(I1)},r>2(\sqrt{2}+1)}{=\!=\!=\!=\!=\!=\!=} \frac{1}{2\pi i} \int_C \frac{(z+1)^n}{z} \frac{\sqrt{4\frac{z+1}{z^2}+1}-1}{2\frac{z+1}{z^2}} \, dz = \frac{1}{2\pi i} \int_C (z+1)^{n-1} = 0$$

since $(z+1)^{n-1}$ is entire. $\square$

Solution 3: Another approach to this problem is using generating functions following the Snake Oil method presented in [6]

We note that $\binom{x}{m} = 0$ when $m < 0$ or if x is a nonnegative integer $< m$ and use $\displaystyle\sum_k$ to indicate the summation over all integers k.

Using the known generating functions

$$\sum_k \frac{\binom{2k}{k}}{k+1} x^k = \frac{1-(1-4x)^{1/2}}{2x} \tag{GF1}$$

$$\sum_{r \geq 0} \binom{r}{k} x^r = \frac{x^k}{(1-x)^{k+1}}, \qquad k \geq 0 \tag{GF2}$$

we multiply S_n (using the same notation as above) with x^n and sum over $n \geq 0$ to get

$$\sum_{n \geq 0} S_n x^n = \sum_{n \geq 0} x^n \sum_k \binom{n+k}{2k}\binom{2k}{k}\frac{(-1)^k}{k+1} = \sum_k \binom{2k}{k}\frac{(-1)^k}{k+1} x^{-k} \sum_{n \geq 0} \binom{n+k}{2k} x^{n+k}$$

$$= \sum_k \binom{2k}{k}\frac{(-1)^k}{k+1} x^{-k} \sum_{r \geq k} \binom{r}{2k} x^r$$

$$\stackrel{\text{(GF2)}}{=} \frac{1}{1-x} \sum_k \binom{2k}{k}\frac{1}{k+1} \left(\frac{-x}{(1-x)^2}\right)^k$$

$$\stackrel{\text{(GF1)}}{=} -\frac{1-x}{2x}\left(1 - \sqrt{1 + \frac{4x}{(1-x)^2}}\right) = 1$$

which means that $S_0 = 1$ and $S_n = 0$ for $n \geq 1$. As in the above solution the fact that $S_{-n} = S_{n-1}$ solves the problem. $\square$

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[2]the sum is finite since $\binom{n+k}{2k} = 0$ for $k > n$.

Remarks: In [7] p.196, the evaluation of $\sum_{k \geq 0} \binom{n+k}{2k}\binom{2k}{k}\frac{(-1)^k}{k+m+1}$, $\mathbb{Z} \ni m, n \geq 0$ is discussed using properties of the binomial coefficients.

A general result relevant to this problem holds:

If two sequences f_n, c_k, $n, k \geq 0$ are connected by the equations

$$f_n = \sum_k \binom{n+k}{m+2k} c_k, \quad n \geq 0,$$

where $m \geq 0$ is fixed, then their generating functions, F and C respectively, are connected by

$$F(x) = \frac{x^m}{(1-x)^{m+1}} C\left(\frac{x}{(1-x)^2}\right).$$

(See [8] p.64 or [9] for more general results.)

V1-3 Evaluate

$$\sum_{k=0}^{n} (-1)^k \binom{2n+1}{k}(2n+1-2k)^{2j+1}$$

for $j \in \{0, \ldots n-1\}$.

Solution : Omran Kouba, *Higher Institute for Applied Sciences and Technology, Damascus, Syria*

The answer is 0.

Let us denote the considered sum by $T_n(j)$, the change of summation variable $k \leftarrow 2n+1-k$ shows that :

$$T_n(j) = \sum_{k=0}^{n}(-1)^k\binom{2n+1}{k}(2n+1-2k)^{2j+1} = \sum_{k=n+1}^{2n+1}(-1)^k\binom{2n+1}{k}(2n+1-2k)^{2j+1}$$

So,

$$2T_n(j) = \sum_{k=0}^{2n+1}(-1)^k\binom{2n+1}{k}P_j(k)$$

where P_j is the polynomial $P_j(X) = (2n+1-2X)^{2j+1}$. It follows that $T_n(j) = 0$ since $\deg P_j \leq 2n-1 < 2n+1$, according to the Lemma that we have proved in our solution of problem V1-2.

$\square$

Remark: This problem occurred in an attempt to evaluate $\int_0^{+\infty} \left(\frac{\sin x}{x}\right)^{2n+1}$ for n a positive integer using complex analysis and has been discussed at the Greek forum www.mathematica.gr: (see www.mathematica.gr/forum/viewtopic.php?f=111&t=11481) where two more solutions have been given for the Lemma Omran Kouba proved here in V1-**2** by Demetres Christofides.

V1-4 For $\mu > 0$, show that

$$(\ln t)^{1/\mu} \sum_{k=1}^{+\infty} \frac{1}{t^{(2k-1)\mu}} = \frac{\Gamma(1/\mu)}{2\mu} + \mathcal{O}\left((t-1)^{1/\mu}\right) \qquad (t \to 1^+).$$

where Γ denotes the Gamma function.

Solution 1: Omran Kouba, *Higher Institute for Applied Sciences and Technology, Damascus, Syria*

Let $f_\mu(x) = \exp(-x^\mu)$ for $x \geq 0$. Since f_μ is clearly decreasing we see that, for $k \geq 1$ and $x > 0$, we have,

$$2xf_\mu((2k-1)x) \geq \int_{(2k-1)x}^{(2k+1)x} f_\mu(t)\, dt \geq 2xf_\mu((2k+1)x)$$

So, if the sum $\displaystyle\sum_{k=1}^{\infty} f_\mu((2k-1)x)$ is denoted by $S_\mu(x)$ we have

$$2xS_\mu(x) \geq \int_x^\infty f_\mu(t)\, dt \geq 2x(S_\mu(x) - e^{-x^\mu}) \tag{1}$$

But, using the change of variables $t = u^{1/\mu}$ we see that

$$\int_0^\infty f_\mu(t)\, dt = \int_0^\infty \exp(-t^\mu)\, dt = \frac{1}{\mu}\int_0^\infty u^{\mu^{-1}-1} e^{-u}\, du = \frac{\Gamma(1/\mu)}{\mu}$$

So, (1) is equivalent to

$$2xe^{-x^\mu} - \int_0^x f_\mu(t) \geq 2xS_\mu(x) - \frac{\Gamma(1/\mu)}{\mu} \geq -\int_0^x f_\mu(t)\, dt \tag{2}$$

Finally, since f_μ is continuous at 0 we have $\displaystyle\int_0^x f_\mu(t)\, dt = \mathcal{O}(x)$ for $x \to 0^+$. Thus, from (2) we conclude that

$$xS_\mu(x) = \frac{\Gamma(1/\mu)}{2\mu} + \mathcal{O}(x) \qquad (x \to 0^+).$$

Now, setting $t = e^{x^\mu}$ we see that $x = (\ln t)^{1/\mu}$ and the result above is equivalent to

$$(\ln t)^{1/\mu} \sum_{k=1}^{\infty} \frac{1}{t^{(2k-1)\mu}} = \frac{\Gamma(1/\mu)}{2\mu} + \mathcal{O}\left((t-1)^{1/\mu}\right) \qquad (t \to 1^+).$$

which is the desired conclusion. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Solution 2: A.Kotronis

The (first) form of Euler MacLaurin summation formula (see [4] p.117) states that

If $: [a, b] \to \mathbb{R}$ where $a, b \in \mathbb{Z}$ is a continuously differentiable function, then

$$\sum_{k=a}^{b} f(k) = \int_a^b f(x)\,dx + \frac{f(b)+f(a)}{2} + \int_a^b \left(\{x\} - \frac{1}{2}\right) f'(x)\,dx$$

is valid

We apply the above to $f_t(x) = t^{-(2x-1)^\mu}$ with $f_t'(x) = -2\mu \ln t (2x-1)^{\mu-1} f_t(x) < 0$ at first on $[1,n]$ and then letting $n \to +\infty$ to get

$$\sum_{k=1}^{+\infty} \frac{1}{t^{(2k-1)^\mu}} = \int_1^{+\infty} t^{-(2x-1)^\mu}\,dx + \frac{1}{2t} + \int_1^{+\infty} \left(\{x\} - \frac{1}{2}\right) f_t'(x)\,dx.$$

Due to the constant sign of f_t', the last integral is in absolute value $\leq \frac{c}{t} = \mathcal{O}(1)$. For the first integral, it is

$$\int_1^{+\infty} t^{-(2x-1)^\mu}\,dx \xxrightarrow{(2x-1)^\mu \ln t = u} \frac{1}{2\mu(\ln t)^{1/\mu}} \left(\Gamma(1/\mu) - \int_0^{\ln t} u^{1/\mu - 1} e^{-u}\,du\right)$$

$$= \frac{\Gamma(1/\mu)}{2\mu(\ln t)^{1/\mu}} + \mathcal{O}(1),$$

since $\dfrac{1}{2\mu(\ln t)^{1/\mu}} \displaystyle\int_0^{\ln t} u^{1/\mu - 1} e^{-u}\,du \xrightarrow{\text{DLH}} \mu$. But $(\ln t)^{1/\mu} = \mathcal{O}\left((t-1)^{1/\mu}\right)$ so, collecting the above and multiplying by $(\ln t)^{1/\mu}$ we get the result. $\square$

Moubinool Omarjee, Lycée Henri IV, Paris proved that

$$\lim_{t \to 1+} (\ln t)^{1/\mu} \sum_{k=1}^{+\infty} t^{-(2k-1)^\mu} = \frac{\Gamma(1/\mu)}{2\mu} \qquad \text{for} \quad \mu > 0.$$

Remark by the editor: This is a slight refinement of problem 10321 [1993, 590] of American Mathematical Monthly where it was asked to be proved that

$$\lim_{t \to 1+} (\ln t)^{1/\mu} \sum_{k=1}^{+\infty} t^{-(2k-1)^\mu} = \frac{\Gamma(1/\mu)}{2\mu} \qquad \text{for} \quad \mu > 0.$$

V1-**5** Compute the following limits, if they exist:

1. $\displaystyle\lim_{s \to +\infty} \frac{1}{\ln s} \int_0^{+\infty} \frac{e^{-\frac{x}{s} - \frac{1}{x}}}{x}\,dx$ and

2. $\displaystyle\lim_{s \to +\infty} \left(\int_0^{+\infty} \frac{e^{-\frac{x}{s} - \frac{1}{x}}}{x}\,dx - \ln s\right).$

Solution 1: Omran Kouba, *Higher Institute for Applied Sciences and Technology, Damascus, Syria*

For $s > 0$, let $F(s)$ be defined by

$$F(s) = \int_0^{+\infty} \frac{e^{-\frac{x}{s} - \frac{1}{x}}}{x} \, dx$$

The convergence of the integral defining $F(s)$ is straightforward. Moreover,

$$F(s) = \underbrace{\int_0^{\sqrt{s}} \frac{e^{-\frac{x}{s} - \frac{1}{x}}}{x} \, dx}_{x = \sqrt{s}\,t} + \underbrace{\int_{\sqrt{s}}^{+\infty} \frac{e^{-\frac{x}{s} - \frac{1}{x}}}{x} \, dx}_{x = \sqrt{s}/t}$$

$$= 2\int_0^1 e^{-\frac{1}{\sqrt{s}}\left(t + \frac{1}{t}\right)} \, dt = 2\int_0^1 e^{-\frac{2}{\sqrt{s}}\varphi(t)} \, dt$$

where $\varphi(t) = \dfrac{1}{2}\left(t + \dfrac{1}{t}\right)$. Now, the change of variables $u = \varphi(t)$ proves that

$$F(s) = 2\int_1^\infty \frac{e^{-\frac{2}{\sqrt{s}}u}}{\sqrt{u^2 - 1}} \, du = 2G\left(\frac{2}{\sqrt{s}}\right) \tag{1}$$

where,

$$G(\lambda) = \int_1^\infty \frac{e^{-\lambda u}}{\sqrt{u^2 - 1}} \, du \tag{2}$$

Now, for $\lambda \in (0, 1)$ we have

$$G(\lambda) = \int_1^\infty \frac{e^{-\lambda u}}{\sqrt{u^2 - 1}} \, du = \int_\lambda^\infty \frac{e^{-v}}{\sqrt{v^2 - \lambda^2}} \, dv \qquad \text{(change of variables } v = \lambda u\text{)}$$

$$= \int_\lambda^1 \frac{e^{-v}}{\sqrt{v^2 - \lambda^2}} \, dv + \int_1^\infty \frac{e^{-v}}{\sqrt{v^2 - \lambda^2}} \, dv$$

$$= \int_\lambda^1 \frac{1}{\sqrt{v^2 - \lambda^2}} \, dv + \int_\lambda^1 \frac{e^{-v} - 1}{\sqrt{v^2 - \lambda^2}} \, dv + \int_1^\infty \frac{e^{-v}}{\sqrt{v^2 - \lambda^2}} \, dv$$

Thus,

$$G(\lambda) = \ln(1 + \sqrt{1 - \lambda^2}) - \ln\lambda + \int_\lambda^1 \frac{e^{-v} - 1}{v} \, dv + \int_1^\infty \frac{e^{-v}}{v} \, dv - H(\lambda) + K(\lambda), \tag{3}$$

where

$$H(\lambda) = \int_\lambda^1 \left(\frac{1}{\sqrt{v^2 - \lambda^2}} - \frac{1}{v}\right)(1 - e^{-v}) \, dv \quad \text{and} \quad K(\lambda) = \int_1^\infty \left(\frac{1}{\sqrt{v^2 - \lambda^2}} - \frac{1}{v}\right) e^{-v} \, dv$$

Now,

$$H(\lambda) = \int_\lambda^1 \left(\frac{v}{\sqrt{v^2 - \lambda^2}} - 1\right) \frac{1 - e^{-v}}{v} \, dv$$

Using $0 < 1 - e^{-v} \le v$ for $v \in (0, 1)$ we conclude that

$$0 \le H(\lambda) \le \int_\lambda^1 \left(\frac{v}{\sqrt{v^2 - \lambda^2}} - 1\right) dv = \sqrt{1 - \lambda^2} - 1 + \lambda \le \lambda$$

On the other hand

$$K(\lambda) = \lambda^2 \int_1^\infty \frac{e^{-v}}{v\sqrt{v^2 - \lambda^2}(v + \sqrt{v^2 - \lambda^2})} \, dv$$

So,

$$0 \leq K(\lambda) \leq \lambda^2 \int_1^\infty \frac{e^{-v}}{\sqrt{1-\lambda^2}(1+\sqrt{1-\lambda^2})}\, dv \leq \frac{\lambda^2}{1-\lambda^2}$$

In particular, we have shown that $K(\lambda) - H(\lambda) = \mathcal{O}(\lambda)$ as $\lambda \to 0^+$. Using this in (3) we see that

$$G(\lambda) = \ln\left(\frac{2}{\lambda}\right) + \int_0^1 \frac{e^{-v}-1}{v}\, dv + \int_1^\infty \frac{e^{-v}}{v}\, dv + \mathcal{O}(\lambda) \qquad (\lambda \to 0^+)$$

Finally, it is well-known that

$$\int_0^1 \frac{1-e^{-v}}{v}\, dv - \int_1^\infty \frac{e^{-v}}{v}\, dv = \gamma$$

where γ is Euler's constant. Thus

$$G(\lambda) = \ln\left(\frac{2}{\lambda}\right) - \gamma + \mathcal{O}(\lambda) \qquad (\lambda \to 0^+)$$

Going back to (1) we see that

$$F(s) = \ln s - 2\gamma + \mathcal{O}\left(\frac{1}{\sqrt{s}}\right) \qquad (s \to +\infty)$$

This implies that

$$\lim_{s\to\infty} \frac{F(s)}{\ln s} = 1 \qquad \text{and} \qquad \lim_{s\to\infty}(F(s) - \ln s) = -2\gamma.$$

and achieves the solution of the problem.   □

Solution 2:

The integral is easily seen to be convergent for $s > 0$. Now

$$\int_0^{+\infty} \frac{e^{-\frac{x}{s}-\frac{1}{x}}}{x}\, dx = \int_0^{\sqrt{s}} \frac{e^{-\frac{x}{s}-\frac{1}{x}}}{x}\, dx + \int_{\sqrt{s}}^{+\infty} \frac{e^{-\frac{x}{s}-\frac{1}{x}}}{x}\, dx$$

and making the change of variables $x = s/t$ for the first integral we get

$$\int_0^{+\infty} \frac{e^{-\frac{x}{s}-\frac{1}{x}}}{x}\, dx = 2 \int_{\sqrt{s}}^{+\infty} \frac{e^{-\frac{x}{s}-\frac{1}{x}}}{x}\, dx := 2F(s).$$

Since

$$e^{-\frac{1}{x}} = 1 + \mathcal{O}(x^{-1}), \qquad x \geq \sqrt{s},$$

we get

$$\frac{e^{-\frac{x}{s}-\frac{1}{x}}}{x} = \frac{e^{-\frac{x}{s}}}{x} + \mathcal{O}\left(\frac{e^{-\frac{x}{s}}}{x}\right), \qquad x \geq \sqrt{s}, s \to +\infty, \text{ so}$$

$$F(s) = \int_{\sqrt{s}}^{+\infty} \frac{e^{-\frac{x}{s}-\frac{1}{x}}}{x}\,dx = \int_{\sqrt{s}}^{+\infty} \frac{e^{-\frac{x}{s}}}{x}\,dx + \mathcal{O}\left(\int_{\sqrt{s}}^{+\infty} \frac{e^{-\frac{x}{s}}}{x}\,dx\right)$$

$$= \int_{\sqrt{s}}^{+\infty} \frac{e^{-\frac{x}{s}}}{x}\,dx + \mathcal{O}\left(\int_{\sqrt{s}}^{+\infty} x^{-2}\,dx\right)$$

$$= \int_{\sqrt{s}}^{+\infty} \frac{e^{-\frac{x}{s}}}{x}\,dx + \mathcal{O}\left(s^{-1/2}\right)$$

Furthermore,

$$\int_{\sqrt{s}}^{+\infty} \frac{e^{-\frac{x}{s}}}{x}\,dx \overset{x=st}{=\!=\!=\!=} \int_{s^{-1/2}}^{+\infty} \frac{e^{-t}}{t}\,dt = e^{-t}\ln t\,\Big|_{s^{-1/2}}^{+\infty} + \int_{s^{-1/2}}^{+\infty} e^{-t}\ln t\,dt$$

$$= \frac{1}{2}e^{-s^{-1/2}}\ln s + \underbrace{\int_{0}^{+\infty} e^{-t}\ln t\,dt}_{=\gamma} - \int_{0}^{s^{-1/2}} e^{-t}\ln t\,dt$$

$$= \frac{\ln s}{2} - \gamma - \int_{0}^{s^{-1/2}} e^{-t}\ln t\,dt + \mathcal{O}\left(\frac{\ln s}{s^{1/2}}\right)$$

Additionally, $\displaystyle\lim_{s\to+\infty} \frac{\int_{0}^{s^{-1/2}} e^{-t}\ln t\,dt}{\int_{0}^{s^{-1/2}} \ln t\,dt} \overset{\text{DLH}}{=} \lim_{s\to+\infty} \frac{e^{-s^{-1/2}}\ln s}{\ln s} = 1$ so

$$\int_{0}^{s^{-1/2}} e^{-t}\ln t\,dt = \mathcal{O}\left(\int_{0}^{s^{-1/2}} \ln t\,dt\right) = \mathcal{O}\left(\frac{\ln s}{s^{1/2}}\right)$$

and collecting we get

$$\int_{0}^{+\infty} \frac{e^{-\frac{x}{s}-\frac{1}{x}}}{x}\,dx = \ln s - 2\gamma + \mathcal{O}\left(\frac{\ln s}{s^{1/2}}\right)$$

which solves the problem.   □

Remark: This is a refinement of problem 6.3 p.79 of [5] which discusses the first limit and the second solution follows the steps of the proof presented there.

V1-**6** Show that the equation $ye^y = x$ with $y(0) = 0$ defines a function $y = y(x)$ in $[0, +\infty)$.

For this function, $y(x)$, compute the following limits, if they exist :

1. $\displaystyle\lim_{x\to+\infty} \frac{y(x)}{\ln x}$,

2. $\displaystyle\lim_{x\to+\infty} \frac{y(x) - \ln x}{\ln(\ln x)}$,

3. $\displaystyle\lim_{x\to+\infty} (y(x) - \ln x + \ln(\ln x))\frac{\ln x}{\ln(\ln x)}$.

Solution: Moubinool Omarjee, *Lycée Henri IV, Paris*

The function $f(t) = te^t$ is strictly increasing on $[0, +\infty[$ so for any $x \geq 0$ there exist a unique solution y such that $ye^y = x$ with $y(0) = 0$. Also, clearly $\lim\limits_{x \to +\infty} y(x) = +\infty$. Now

$$\ln(y(x)) + y(x) = \ln x \tag{1}$$

gives

$$\lim_{x \to +\infty} \frac{y(x)}{\ln x} = 1,^3 \qquad \text{or}$$

$$y(x) = \ln x + o(\ln x). \tag{2}$$

Plugging (2) in (1) we get

$$y(x) + \ln(\ln x + o(\ln x)) = \ln x, \qquad \text{so}$$

$$y(x) + \ln(\ln x) + \ln(1 + o(1)) = \ln x$$

which gives

$$\lim_{x \to +\infty} \frac{y(x) - \ln x}{\ln(\ln x)} = -1, \qquad \text{or}$$

$$y(x) = \ln x - \ln(\ln x) + o(\ln(\ln x)).$$

Plugging again the above in (1) gives

$$y(x) + \ln(\ln x - \ln(\ln x) + o(\ln(\ln x))) = \ln x, \qquad \text{so}$$

$$y(x) + \ln(\ln x) + \ln\left(1 - \frac{\ln(\ln x)}{\ln x} + o\left(\frac{\ln(\ln x)}{\ln x}\right)\right) = \ln x, \qquad \text{so}$$

$$y(x) + \ln(\ln x) - \frac{\ln(\ln x)}{\ln x} + o\left(\frac{\ln(\ln x)}{\ln x}\right) = \ln x$$

and this gives

$$\lim_{x \to +\infty} \frac{y(x) - \ln x + \ln(\ln x)}{\ln(\ln x)} \ln x = 1.$$

$\square$

Also solved by Omran Kouba, *Higher Institute for Applied Sciences and Technology, Damascus, Syria*

Remarks: This problem is discussed in [1] p.25 as well as in [2] p.206 and appears as a problem in [3] p.63. It has also been discussed at the Greek forum www.mathematica.gr: (see http://www.mathematica.gr/forum/viewtopic.php?f=55&t=26942 and http://www.mathematica.gr/forum/viewtopic.php?f=9&t=18098).

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[3]Editor's note: For x big enough, (1) gives $\frac{y(x)}{\ln x} = \left(1 + \frac{\ln(y(x))}{y(x)}\right)^{-1}$ and since $\lim\limits_{x \to +\infty} y(x) = +\infty$ we get $\lim\limits_{x \to +\infty} \frac{y(x)}{\ln x} = 1$.

V1-7 Let a_n the sequence defined by $a_n = n(n-1)a_{n-1} + \dfrac{n(n-1)^2}{2}a_{n-2}$ for $n \geq 3$ and $a_1 = 0, a_2 = 1$.

    1. Show that $\displaystyle\lim_{n\to+\infty} \frac{e^{2n}a_n}{n^{2n+1/2}} = 2\sqrt{\frac{\pi}{e}}$, and

    2. compute $\displaystyle\lim_{n\to+\infty} n\left(\frac{e^{2n}a_n}{n^{2n+1/2}} - 2\sqrt{\frac{\pi}{e}}\right)$ if it exists.

Solution: Omran Kouba, *Higher Institute for Applied Sciences and Technology, Damascus, Syria*
For $n \geq 0$, let λ_n be defined by

$$\lambda_n = \frac{n!}{\sqrt{\pi}} \int_0^\infty \left(t - \frac{1}{2}\right)^n \frac{e^{-t}}{\sqrt{t}}\, dt$$

For $n \geq 2$ we have

$$n(n-1)\lambda_{n-1} + \frac{n(n-1)^2}{2}\lambda_{n-2} = \frac{n!}{\sqrt{\pi}} \int_0^\infty (n-1)\left(t - \frac{1}{2}\right)^{n-2} \sqrt{t}\,e^{-t}\, dt$$

$$= \frac{n!}{\sqrt{\pi}} \left[\left(t - \frac{1}{2}\right)^{n-1} \sqrt{t}\,e^{-t}\right]_0^\infty$$

$$- \frac{n!}{\sqrt{\pi}} \int_0^\infty \left(t - \frac{1}{2}\right)^{n-1} \left(\frac{1}{2\sqrt{t}} - \sqrt{t}\right) e^{-t}\, dt$$

$$= \frac{n!}{\sqrt{\pi}} \int_0^\infty \left(t - \frac{1}{2}\right)^n \frac{e^{-t}}{\sqrt{t}}\, dt = \lambda_n$$

So, we have shown that

$$\forall\, n \geq 2, \quad \lambda_n = n(n-1)\lambda_{n-1} + \frac{n(n-1)^2}{2}\lambda_{n-2}. \tag{1}$$

Moreover, if Γ is the well-known eulerian gamma function then, $\lambda_0 = \dfrac{1}{\sqrt{\pi}}\Gamma\left(\dfrac{1}{2}\right) = 1$, and

$$\lambda_1 = \frac{1}{\sqrt{\pi}}\left(\Gamma\left(\frac{3}{2}\right) - \frac{1}{2}\Gamma\left(\frac{1}{2}\right)\right) = 0 = a_1$$

Using (1) we see that $\lambda_2 = 1 = a_2$. Thus, the sequence $(\lambda_n)_{n\geq 1}$ satisfies, the same recurrence relation as the sequence $(a_n)_{n\geq 1}$, and the it has the same initial conditions. This proves that $a_n = \lambda_n$ for every $n \geq 1$. So we have proved that

$$\forall\, n \geq 1, \quad a_n = \frac{n!}{\sqrt{\pi}} \int_0^\infty \left(t - \frac{1}{2}\right)^n \frac{e^{-t}}{\sqrt{t}}\, dt. \tag{2}$$

Now, let us define b_n and c_n by

$$b_n = \frac{n!}{\sqrt{\pi}} \int_0^{1/2} \left(t - \frac{1}{2}\right)^n \frac{e^{-t}}{\sqrt{t}}\, dt = \frac{(-1)^n n!}{2^n\sqrt{2\pi}} \int_0^1 (1-u)^n u^{-1/2} e^{-u/2}\, du \qquad (u \leftarrow t/2)$$

$$c_n = \frac{n!}{\sqrt{\pi}} \int_{1/2}^\infty \left(t - \frac{1}{2}\right)^n \frac{e^{-t}}{\sqrt{t}}\, dt = \frac{n!}{\sqrt{\pi e}} \int_0^\infty u^n \frac{e^{-u}}{\sqrt{u + 1/2}}\, du \qquad (u \leftarrow t - 1/2)$$

so that $a_n = b_n + c_n$. But,

$$\int_0^1 (1-u)^n u^{-1/2} e^{-u/2}\, du = \frac{1}{\sqrt{n}} \int_0^n \left(1 - \frac{v}{n}\right)^n v^{-1/2} e^{-\frac{v}{2n}}\, dv$$

$$\leq \frac{1}{\sqrt{n}} \int_0^n \left(1 - \frac{v}{n}\right)^n v^{-1/2}\, dv$$

$$\leq \frac{1}{\sqrt{n}} \int_0^n v^{-1/2} e^{-v}\, dv \qquad\qquad (\text{since } 1 - x \leq e^{-x})$$

$$\leq \frac{1}{\sqrt{n}} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{\sqrt{n}}$$

Thus $|b_n| \leq \dfrac{n!}{2^n \sqrt{2n}}$. In particular,

$$b_n = \mathcal{O}\left(\left(\frac{n}{2e}\right)^n\right). \tag{3}$$

Now, let us come to c_n. Note that for $x > 0$ we have

$$\frac{1}{x^2}\left(\frac{x}{2} - 1 + \frac{1}{\sqrt{1+x}}\right) = \frac{1}{\sqrt{1+x}\,(1+\sqrt{1+x})} + \frac{1}{\sqrt{1+x}\,(1+\sqrt{1+x})^2} \in \left[0, \frac{3}{8}\right]$$

So, taking $x = 1/(2u)$, we see that

$$0 \leq 4u^2 \left(\frac{1}{4u} - 1 + \frac{\sqrt{u}}{\sqrt{u+1/2}}\right) \leq \frac{3}{8}$$

or equivalently, for $u > 0$

$$0 \leq \frac{1}{\sqrt{u+1/2}} - \frac{1}{u^{1/2}} + \frac{1}{4u^{3/2}} \leq \frac{3}{32u^{5/2}}$$

This implies that

$$0 \leq \int_0^\infty u^n \frac{e^{-u}}{\sqrt{u+1/2}}\, du - \Gamma\left(n + \frac{1}{2}\right) + \frac{1}{4}\Gamma\left(n - \frac{1}{2}\right) \leq \frac{3}{32}\Gamma\left(n - \frac{3}{2}\right)$$

But $\Gamma\left(n - \dfrac{1}{2}\right) = \dfrac{1}{n - 1/2}\Gamma\left(n + \dfrac{1}{2}\right)$ and $\Gamma\left(n - \dfrac{3}{2}\right) = \dfrac{1}{(n-1/2)(n-3/2)}\Gamma\left(n + \dfrac{1}{2}\right)$, so

$$\int_0^\infty u^n \frac{e^{-u}}{\sqrt{u+1/2}}\, du = \Gamma\left(n + \frac{1}{2}\right)\left(1 - \frac{1}{4n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right).$$

and, since $n!\,\Gamma\left(n + \dfrac{1}{2}\right)) = \dfrac{(2n)!\sqrt{\pi}}{2^{2n}}$ we conclude that

$$c_n = \frac{(2n)!}{\sqrt{e}\, 2^{2n}}\left(1 - \frac{1}{4n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right).$$

But, by stirling's expansion we know that $(2n)! = 2(2n)^{2n}e^{-2n}\sqrt{\pi n}\left(1 + \dfrac{1}{24n} + \mathcal{O}\left(\dfrac{1}{n^2}\right)\right)$, thus

$$\frac{e^{2n}}{n^{2n+1/2}}c_n = 2\sqrt{\frac{\pi}{e}}\left(1 + \frac{1}{24n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right)\left(1 - \frac{1}{4n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right)$$

$$= 2\sqrt{\frac{\pi}{e}}\left(1 - \frac{5}{24n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right)$$

By (3), clearly $\dfrac{e^{2n}}{n^{2n+1/2}}b_n = \mathcal{O}\left(\dfrac{1}{n^2}\right)$ so

$$\frac{e^{2n}}{n^{2n+1/2}}a_n = 2\sqrt{\frac{\pi}{e}}\left(1 - \frac{5}{24n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right).$$

This proves part 1, and shows that the limit in part 2 exists and is equal to $-\dfrac{5}{12}\sqrt{\dfrac{\pi}{e}}$. $\square$

Solution: A.KOTRONIS

At first we set $b_n = \dfrac{2a_{n+2}}{(n+1)!(n+2)!}$ which transforms the given recurrence to

$$2(n+2)b_{n+2} = 2(n+2)b_{n+1} + b_n, \qquad n \geq 0 \qquad b_0 = b_1 = 1.$$

Now we calculate the generating function $f(z) := \displaystyle\sum_{n \geq 0} b_n z^n$ of b_n. We multiply the recurrence with z^n and sum for $n \geq 0$ to get

$$\frac{2}{z}\sum_{n \geq 2}nb_n z^{n-1} = 2\sum_{n \geq 1}nb_n z^{n-1} + \frac{2}{z}\sum_{n \geq 1}b_n z^n + \sum_{n \geq 0}b_n z^n, \qquad \text{or}$$

$$\frac{2}{z}(f'(z) - 1) = 2f'(z) + \frac{2}{z}(f(z) - 1) + f(z), \qquad \text{or}$$

$$f'(z) + \left(\frac{1}{2} - \frac{3}{2(1-z)}\right)f(z) = 0$$

which gives, using that $b_0 = 1$, that $f(z) = e^{-z/2}(1-z)^{-3/2}$.

Now, since $e^{-z/2}$ is entire, expanding it around 1 we have that

$$f(z) := e^{-1/2}(1-z)^{-3/2} + \frac{e^{-1/2}}{2}(1-z)^{-1/2} + (1-z)^{1/2}g(z) \tag{I}$$

where $g(z)$ is entire and hence analytic on $|z| < R$ with $R > 1$, so, setting

$$c_n := [z^n]\{(1-z)^{1/2}\} \quad \text{and} \quad d_n := [z^n]\{g(z)\},$$

we have that $d_n = \mathcal{O}(\varepsilon^n)$ for some $0 < \varepsilon < 1$ and furthermore, since by Stirling's formula

$$c_n = (-1)^n \binom{1/2}{n} = \binom{-1/2 + n - 1}{n} = \binom{n - 3/2}{n} = \frac{\Gamma(n - 1/2)}{\Gamma(-1/2)\Gamma(n + 1)} = \mathcal{O}(n^{-3/2}),$$

we get

$$\left| [z^n] \left\{ (1 - z)^{1/2} g(z) \right\} \right| = \left| \sum_{0 \leq k \leq n} c_k d_{n-k} \right| \leq \left| \sum_{0 \leq k \leq n/2} c_k d_{n-k} \right| + \left| \sum_{n/2 < k \leq n} c_k d_{n-k} \right|$$

$$\leq \max_{0 \leq k \leq n/2} |c_k| \sum_{n/2 \leq k \leq n} \varepsilon^k + \max_{n/2 \leq k \leq n} |c_k| \sum_{0 \leq k \leq n/2} \varepsilon^k$$

$$= \mathcal{O}\left(\max\{1, n^{-3/2}\} \right) \mathcal{O}(\varepsilon^{n/2}) + \mathcal{O}(n^{-3/2})\mathcal{O}(1)$$

$$= \mathcal{O}(n^{-3/2}).$$

On account of the above, from (I), it is

$$b_n = [z^n]\{f(z)\} = e^{-1/2}[z^n]\{(1 - z)^{-3/2}\} + \frac{e^{-1/2}}{2}[z^n]\{(1 - z)^{-1/2}\} + [z^n]\{(1 - z)^{1/2}g(z)\}$$

$$= e^{-1/2}\frac{\Gamma(n + 3/2)}{\Gamma(3/2)\Gamma(n + 1)} + \frac{e^{-1/2}}{2}\frac{\Gamma(n + 1/2)}{\Gamma(1/2)\Gamma(n + 1)} + \mathcal{O}(n^{-3/2})$$

so, applying Stirling once more:

$$\frac{e^{2n}}{n^{2n+1/2}}a_n = \frac{e^{2n}}{n^{2n+1/2}}\frac{(n-1)!n!}{2}b_{n-2}$$

$$= \frac{e^{2n}}{n^{2n+1/2}}\frac{(n-1)!n!}{2}\left(e^{-1/2}\frac{\Gamma(n - 1/2)}{\Gamma(3/2)\Gamma(n - 1)} + \frac{e^{-1/2}}{2}\frac{\Gamma(n - 3/2)}{\Gamma(1/2)\Gamma(n - 1)} + \mathcal{O}(n^{-3/2}) \right)$$

$$= 2\sqrt{\frac{\pi}{e}} - \frac{5}{12}\sqrt{\frac{\pi}{e}}n^{-1} + \mathcal{O}(n^{-2})$$

which solves the problem.   □

Remark The above problem is discussed at [10] (Problem 79-5 p.350) and the second solution uses the techniques presented at [6] (see chapter 5 "Analytic and Asymptotic Methods").

V1-8 If $a_1, \ldots, a_n, b_1, \ldots, b_m, b \in \mathbb{R}$ with $\displaystyle\sum_{k=1}^{n} |a_k| + \sum_{k=1}^{m} |b_k| < b$, evaluate

$$\int_0^{+\infty} \frac{\sin(bx)}{x} \prod_{k=1}^{n} \frac{\sin(a_k x)}{x} \prod_{k=1}^{m} \cos(b_k x)\, dx.$$

Solution: OMRAN KOUBA, *Higher Institute for Applied Sciences and Technology, Damascus, Syria*

Recall that $\int_0^\infty \frac{\sin(x)}{x}\,dx = \pi/2$. It follows that

$$\int_0^\infty \frac{\sin(bx)}{x}\,dx = \text{sgn}(b)\frac{\pi}{2}. \tag{1}$$

Now suppose that β and b are real numbers such that $|\beta| < b$. Using (1) and the fact that both $b - \beta$ and $b + \beta$ are positive we see that

$$\int_0^\infty \frac{\sin(bx)}{x}\cos(\beta x)\,dx = \frac{1}{2}\int_0^\infty \frac{\sin((b+\beta)x) + \sin((b-\beta)x)}{x}\,dx = \frac{\pi}{2}. \tag{2}$$

Now consider real numbers $\alpha_1, \ldots, \alpha_p$ and b such that $\sum_{k=1}^p |\alpha_k| < b$. Note that

$$\prod_{k=1}^p \cos(\alpha_k x) = \frac{1}{2^p}\prod_{k=1}^p \left(e^{i\alpha_k x} + e^{-i\alpha_k x}\right) = \frac{1}{2^p}\sum_{A\subset\mathbb{N}_p} e^{i(\alpha_A - \alpha_{A'})x},$$

where $\mathbb{N}_p = \{1, 2, \ldots, p\}$, $A' = \mathbb{N}_p \setminus A$ and $\alpha_B = \sum_{k\in B}\alpha_k$. So,

$$\prod_{k=1}^p \cos(\alpha_k x) = \frac{1}{2^p}\sum_{A\subset\mathbb{N}_{p-1}}\left(e^{i(\alpha_A - \alpha_{A'})x} + e^{i(\alpha_{A'} - \alpha_A)x}\right) = \frac{1}{2^{p-1}}\sum_{A\subset\mathbb{N}_{p-1}}\cos(\beta_A x)$$

where $\beta_A = \alpha_A - \alpha_{A'}$. The assumption implies that $|\beta_A| < b$ for every subset A of $\mathbb{N}_{p-1}$. So, using (2) we obtain

$$\int_0^\infty \frac{\sin(bx)}{x}\prod_{k=1}^p \cos(\alpha_k x)\,dx = \frac{1}{2^{p-1}}\sum_{A\subset\mathbb{N}_{p-1}}\int_0^\infty \frac{\sin(bx)}{x}\cos(\beta_A x)\,dx = \frac{\pi}{2}. \tag{3}$$

Now, consider real numbers $a_1, \ldots, a_n, b_1, \ldots, b_m, b$ such that $\sum_{k=1}^n |a_k| + \sum_{k=1}^m |b_k| < b$. Applying (3) with $p = n + m$, $|\alpha_k| \le |a_k|$ for $k = 1, \ldots, n$, and $\alpha_{n+k} = b_k$ for $k = 1, \ldots, m$, we obtain

$$\int_0^\infty \frac{\sin(bx)}{x}\prod_{k=1}^n \cos(\alpha_k x)\prod_{k=1}^m \cos(b_k x)\,dx = \frac{\pi}{2}$$

Now, for each $k \in \{1, \ldots, n\}$, we integrate both sides of the above equation with respect to α_k from 0 to a_k, we obtain

$$\int_0^\infty \frac{\sin(bx)}{x}\prod_{k=1}^n \frac{\sin(a_k x)}{x}\prod_{k=1}^m \cos(b_k x)\,dx = \frac{\pi}{2}\prod_{k=1}^n a_k,$$

which is the desired conclusion. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Box$

Remark This problem is discussed in [11] p.41 using complex analysis methods. Specifically, the function $F(z) = \frac{e^{bzi}}{z}\prod_{k=1}^n \frac{\sin(a_k z)}{z}\prod_{k=1}^m \cos(b_k z)$ is integrated along the contour consisting of the

semicircles $\gamma = \{z : |z| = r, \Im mz > 0\}$ and $\Gamma\{z : |z| = R, \Im mz > 0\}$ with $R > r$ and the real segments $[-R, -r]$, $[r, R]$.

A solution has also has been given at the Greek forum www.mathematica.gr: (see http://www.mathe matica.gr/forum/viewtopic.php?f=9&t=7842&start=140 by Kostas Tsouvalas.)

V1-9 Evaluate $\displaystyle\lim_{n \to +\infty} \sum_{k=1}^{+\infty} \frac{(-1)^k}{\sqrt[n]{k^k}}$, if it exists.

(if the limit exists and is a real number ℓ, can we make a better estimate than $\displaystyle\sum_{k=1}^{+\infty} \frac{(-1)^k}{\sqrt[n]{k^k}} = \ell + o(1)$?)

Solution: OMRAN KOUBA, *Higher Institute for Applied Sciences and Technology, Damascus, Syria*

We will prove that $\displaystyle\sum_{k=1}^{+\infty} \frac{(-1)^k}{\sqrt[n]{k^k}} = -\frac{1}{2} + \mathcal{O}\left(\frac{\ln n}{n}\right)$. Indeed, the convergence of the series $\displaystyle\sum_{k=1}^{+\infty} \frac{(-1)^k}{\sqrt[n]{k^k}}$ is ensured by the alternating series convergence test. Let the sum of this series be denoted by G_n, and let us define the function f_n on $[1, +\infty)$ by $f_n(x) = \exp\left(-\frac{1}{n}x \ln x\right)$. Now we have

$$G_n = \sum_{k=1}^{\infty} (-1)^k f_n(k) = \sum_{k=1}^{\infty} (f_n(2k) - f_n(2k-1))$$

$$= -1 + \sum_{k=2}^{\infty} (-1)^k f_n(k) = -1 + \sum_{k=1}^{\infty} (f_n(2k) - f_n(2k+1))$$

Thus,

$$2G_n + 1 = \sum_{k=1}^{\infty} (2f_n(2k) - f_n(2k+1) - f_n(2k-1)) = \sum_{k=1}^{\infty} a_n(k), \tag{1}$$

where,

$$a_n(k) = 2f_n(2k) - f_n(2k+1) - f_n(2k-1). \tag{2}$$

Now, using integration by parts we have

$$\int_0^1 (u-1)(f_n''(2k+u) + f_n''(2k-u))du = \left[(u-1)(f_n'(2k+u) - f_n'(2k-u))\right]_0^1$$

$$- \int_0^1 \left(f_n'(2k+u) - f_n'(2k-u)\right)du$$

$$= 2f_n(2k) - f_n(2k+1) - f_n(2k-1) = a_n(k) \tag{3}$$

So,

$$|a_n(k)| \le \int_0^1 (|f_n''(2k+u)| + |f_n''(2k-u)|)du = \int_{2k-1}^{2k+1} |f_n''(t)|\, dt$$

and using (1) we conclude that

$$|2G_n + 1| \le \sum_{k=1}^{\infty} |a_n(k)| \le \int_1^{+\infty} |f_n''(t)|\, dt \tag{3}$$

Next we suppose that $n > 1$. Clearly,

$$f_n'(x) = -\frac{1}{n}(1 + \ln x)f_n(x),$$

$$f_n''(x) = \frac{1}{n^2}\left((1 + \ln x)^2 - \frac{n}{x}\right)f_n(x),$$

and, if $g(x) = (1 + \ln x)^2 - n/x$ then g is increasing on $[1, +\infty)$ (as sum of increasing functions.) But $g(1) = 1 - n < 0$ and $g(n) = (2 + \ln n)\ln n > 0$, so there exists a unique real number $x_n \in (1, n)$ such that $g(x_n) = 0$. Moreover, $g(x) < 0$ if $x \in [1, x_n)$ and $g(x) > 0$ if $x \in (x_n, +\infty)$. Therefore,

$$\int_1^\infty |f_n''(x)|\,dx = -\int_1^{x_n} f_n''(x)\,dx + \int_{x_n}^\infty f_n''(x)\,dx$$

$$= f_n'(1) + \lim_{x \to +\infty} f_n'(x) - 2f_n'(x_n)$$

$$= -\frac{1}{n} - 2f_n'(x_n) \le \frac{2}{n}(1 + \ln x_n)f_n(x_n).$$

But $1 + \ln x_n = \sqrt{\dfrac{n}{x_n}}$, (since $g(x_n) = 0$,) and $f_n(x_n) < 1$. Therefore, from (3) we conclude that

$$|G_n + \frac{1}{2}| < \frac{1}{\sqrt{nx_n}}. \tag{4}$$

Now, suppose that $n \ge 6$ so that $\ln(\ln n) > 1/2$, it follows that

$$g\left(\frac{n}{\ln^2 n}\right) = (1 + \ln n - 2\ln(\ln n))^2 - \ln^2 n = (1 - 2\ln(\ln n))(1 + 2\ln n - 2\ln(\ln n)) < 0$$

This proves that $x_n > n/\ln^2 n$. So, (4) implies that

$$|G_n + \frac{1}{2}| < \frac{\ln n}{n} \qquad \text{for } n \ge 6.$$

This proves that

$$\sum_{k=1}^{+\infty} \frac{(-1)^k}{\sqrt[n]{k^k}} = -\frac{1}{2} + \mathcal{O}\left(\frac{\ln n}{n}\right)$$

which is the desired conclusion. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Remark by the solver: Numerical experiment show that in fact

$$\sum_{k=1}^{+\infty} \frac{(-1)^k}{\sqrt[n]{k^k}} = -\frac{1}{2} - \frac{\beta}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)$$

where $\beta \approx 0.265214371$. But I couldn't prove this.

Remark by A.Kotronis: The source of this problem is this discussion: http://www.artofproblemsolv ing.com/Forum/viewtopic.php?f=67&t=379317 at the Art Of Problem Solving forum.

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