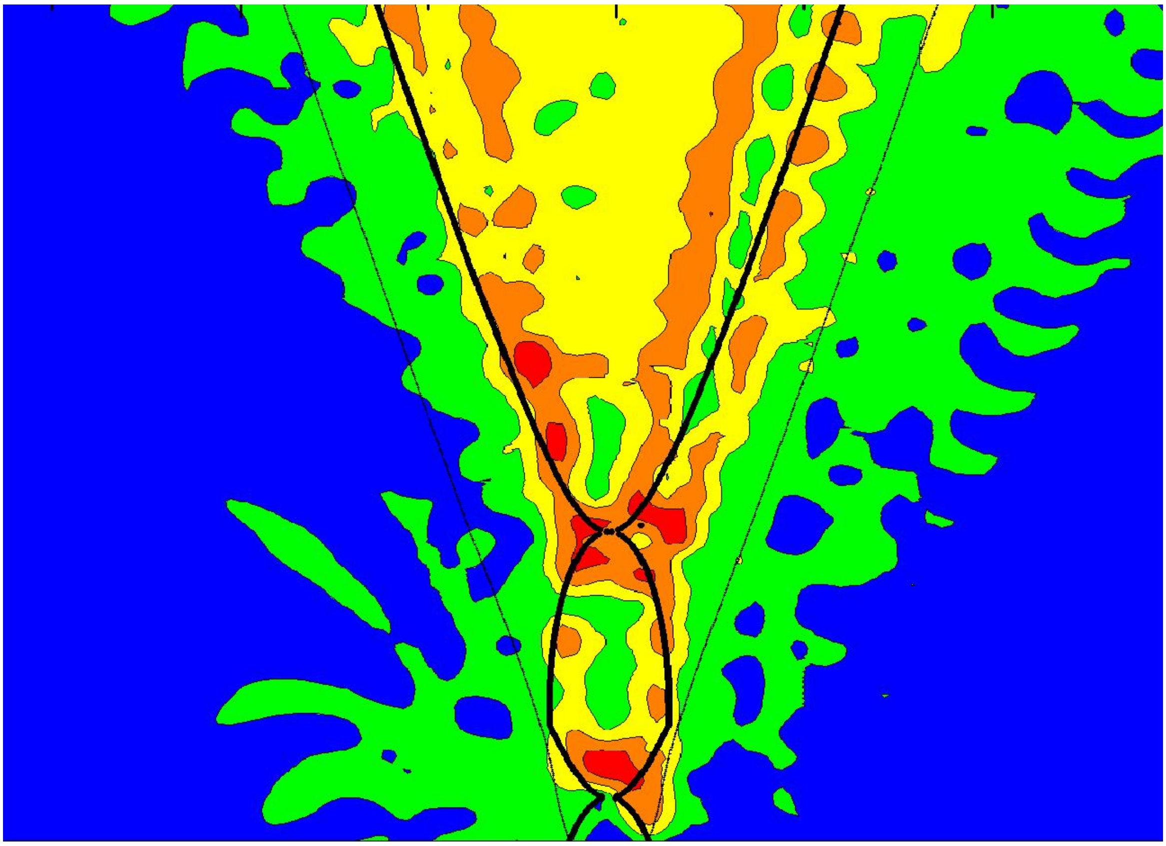
Asymmetry

mathematical journal

Volume 3 March 2013



Picture by M.Miliaresi's graduate thesis "Dynamics of asteroids in resonance with Jupiter".

Editor

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Asymmetry is an electronic mathematical journal which aims at giving the chance to the readers, and the editor himself, to work on interesting mathematical problems or find information about various mathematical topics. The problems presented may either be original or taken from the existing literature or the web. Attempt will be made to be precise as regards the problems' original source. The level of the topics is undergraduate and beyond. Readers are encouraged by the editor to submit proposals and/or solutions to proposed problems. Proposals and solutions are preferred to be in LaTeX format using what is necessary from the preamble presented in `http://akotronismaths.blogspot.gr/p/asymmetry-electronic-mathematical.html`, must be legible and should appear on separate sheets, each indicating the name of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that no solution is available at the time the problem is published. Questions concerning proposals and/or solutions can be sent by e-mail to akotronis@gmail.com.

Editor's note

The editor would like to invite the interested reader to visit the new mathematical forum

www.mathimatikoi.org

This forum is for university level mathematics, in the English language, and was made by a group of Greek mathematicians.

Problems and Solutions

The source of the problems will appear along with the publication of the solutions

The Problems

V3-1 *Proposed by José Luis Díaz–Barrero, Technical University of Catalonia (BARCELONA TECH), Barcelona, Spain.*

Let $a_1, a_2, \ldots, a_n$, be $n \geq 2$ positive real numbers. Prove that

$$\frac{a_1}{a_2 + 3\sqrt[7]{a_1 a_2^6}} + \frac{a_2}{a_3 + 3\sqrt[7]{a_2 a_3^6}} + \ldots + \frac{a_n}{a_1 + 3\sqrt[7]{a_n a_1^6}} \geq \frac{n}{4}.$$

V3-**2** *Proposed by the editor*

If F_n, L_n and T_n are the $n-$th Fibonacci[1], Lucas[2] and Triangular[3] number respectively, show that

$$2F_n^5 + n^2 T_n F_n - L_{n+1} = \begin{cases} 0 \pmod 5 & , n \neq 0 \pmod 5 \\ 2 \pmod 5 & , n = 5 \pmod{20} \\ 1 \pmod 5 & , n = 10 \pmod{20} \\ 3 \pmod 5 & , n = 15 \pmod{20} \\ 4 \pmod 5 & , n = 0 \pmod{20} \end{cases}, \qquad n \geq 1.$$

V3-**3** Evaluate $\displaystyle\sum_{n\geq 1} (-1)^n \frac{\prod_{j=1}^n \left(\frac{3}{2} - j\right)}{(2n+1)n!}$ if it converges.

V3-**4** Let a_n be the sequence defined by $a_{n+1} = a_n + a_n^{-k}, \quad a_1 > 0,\ k \in \mathbb{R}$.

1. Show that, for $k > -1$:

$$a_n = (k+1)^{\frac{1}{k+1}} n^{\frac{1}{k+1}} \left(1 + \frac{k}{2(k+1)^2} \frac{\ln n}{n} + \mathcal{O}\left(n^{-1}\right)\right)$$

2. (*) Can we make a two terms estimate, as in 1., when $k < -1$?

V3-**5** Show that for $n \in \mathbb{N}^*$:

$$\sum_{r\geq 1} \frac{(-1)^r}{r\binom{n+r}{n}} = 2^n \ln 2 + \sum_{k=0}^{n-1} (-1)^{n-k} \binom{n}{k} \frac{2^n - 2^k}{n-k}.$$

V3-**6** Let $F(x) := \displaystyle\int_0^{+\infty} \frac{1}{e^t + xt}\, dt$ for the values of $x \in \mathbb{R}$ for which it can be defined.

1. Find the MacLaurin expansion of $F(x)$ at 0, if it has one, and determine it's radius of convergence.
2. Show that $\displaystyle\lim_{x\to -e^+} (x+e)^{1/2} \int_0^{+\infty} \frac{1}{e^t + xt}\, dt = \pi\sqrt{\frac{2}{e}}$.
3. (*) Examine whether there exists a real number $a < 0$ such that

$$\lim_{x\to -e^+} (x+e)^a \left((x+e)^{1/2} \int_0^{+\infty} \frac{1}{e^t + xt}\, dt - \pi\sqrt{\frac{2}{e}} \right) \in \mathbb{R}^*$$

and, for this real number a, in the case it exists, compute the limit.

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[1] $= \frac{1}{\sqrt{5}} (a^n - b^n)$, where $a = \frac{1+\sqrt{5}}{2}$, $b = \frac{1-\sqrt{5}}{2}$ see `http://en.wikipedia.org/wiki/Fibonacci_number`

[2] $= a^n + b^n$, where $a = \frac{1+\sqrt{5}}{2}$, $b = \frac{1-\sqrt{5}}{2}$, see `http://en.wikipedia.org/wiki/Lucas_number`

[3] $= \frac{n(n+1)}{2}$, see `http://en.wikipedia.org/wiki/Triangular_number`

Solutions

V2-1 Let $f:[-1,1]\to\mathbb{R}$ be an odd and Riemann integrable function such that $\displaystyle\int_0^{2k\pi} x^2 f(\sin x)\,dx \neq 0$ for $k\in\mathbb{N}$. Evaluate

$$\sum_{k\geq 1}\frac{\int_0^{\pi} f(\sin x)\,dx}{\int_0^{2k\pi} x^2 f(\sin x)\,dx}.$$

Solution: Omran Kouba, *Higher Institute for Applied Sciences and Technology, Damascus, Syria*

Let the integral $\displaystyle\int_0^{\pi} f(\sin x)\,dx$ be denoted by α. Note that

$$\begin{aligned}
\int_{2n\pi}^{2(n+1)\pi} x^2 f(\sin x)\,dx &= \int_0^{2\pi}(2n\pi+x)^2 f(\sin x)\,dx\\
&= \int_0^{\pi}(2n\pi+x)^2 f(\sin x)\,dx+\\
&\quad \int_0^{\pi}(2(n+1)\pi-x)^2 f(\sin(2\pi-x))\,dx && (x\leftarrow 2\pi-x.)\\
&= \int_0^{\pi}\left((2n\pi+x)^2-(2(n+1)\pi-x)^2\right)f(\sin x)\,dx && (f \text{ is odd.})\\
&= -4\pi(2n+1)\int_0^{\pi}(\pi-x)f(\sin x)\,dx && (1)
\end{aligned}$$

But the change of variables $x\leftarrow \pi-x$ shows also that

$$\int_0^{\pi}(\pi-x)f(\sin x)\,dx=\int_0^{\pi}xf(\sin x)\,dx$$

and consequently

$$\begin{aligned}
\int_0^{\pi}(\pi-x)f(\sin x)\,dx &= \frac{1}{2}\left(\int_0^{\pi}(\pi-x)f(\sin x)\,dx+\int_0^{\pi}xf(\sin x)\,dx\right)\\
&= \frac{\pi}{2}\int_0^{\pi}f(\sin x)\,dx=\frac{\pi}{2}\alpha.
\end{aligned}$$

So, replacing in (1) we see that $\displaystyle\int_{2n\pi}^{2(n+1)\pi}x^2 f(\sin x)\,dx=-2\pi^2\alpha(2n+1)$. It follows that

$$\int_0^{2k\pi}x^2 f(\sin x)\,dx=-2\pi^2\alpha\sum_{n=0}^{k-1}(2n+1)=-2\pi^2\alpha\,k^2$$

Thus

$$\frac{\int_0^{\pi} f(\sin x)\,dx}{\int_0^{2k\pi} x^2 f(\sin x)\,dx}=-\frac{1}{2\pi^2}\cdot\frac{1}{k^2}$$

and

$$\sum_{k=1}^{\infty}\frac{\int_0^{\pi} f(\sin x)\,dx}{\int_0^{2k\pi} x^2 f(\sin x)\,dx}=-\frac{1}{2\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}=-\frac{1}{12}$$

which is the desired conclusion. $\square$

Remark by the editor:

The above problem, asking to show that $\int_0^{2k\pi} x^2 f(\sin x)\ dx = -2k^2\pi^2 \int_0^{\pi} f(\sin x)\ dx$ in the case that $f : [-1,1] \to \mathbb{R}$ is odd and Riemann integrable appears in [1] p.429 and in the version presented here has been discussed on the Greek forum `www.mathematica.gr` (`www.mathematica.gr/forum/viewtopic.php?f=9&t=5137`) and on the Art of Problem Solving forum (`www.artofproblemsolving.com/Forum/viewtopic.php?f=67&t=333185`).

V2-2 *Proposed by Spyros Kapellides Ioannina Greece*

Let $p(x)$ be a polynomial with real coefficients such that $\{p(n)\} < \frac{1}{n}$, $\forall n \in \mathbb{N}$. Show that $p(n) \in \mathbb{Z}$, $\forall n \in \mathbb{N}$.

$\{\cdot\}$ denotes the fractional part.

Solution: Omran Kouba, *Higher Institute for Applied Sciences and Technology, Damascus, Syria*

The basic ingredient is the following Lemma.

Lemma. *Let* P *be a polynomial with real coefficients, then*

$$\left(\lim_{n\to\infty} e^{2\pi i P(n)} = 1\right) \implies (\forall\, n \in \mathbb{N},\ P(n) \in \mathbb{Z})$$

Proof. We will prove this by induction on the degree d of P.

If $d \leq 0$, then P is constant, that is $P(n) = P(0)$ for every n. Using the hypothesis we obtain $e^{2\pi i P(0)} = 1$, and this implies that $P(0) \in \mathbb{Z}$ which is the desired conclusion, in this case.

Now, consider $d \geq 1$, and suppose that result is true for every real polynomial of degree smaller than d. Now, consider a polynomial P of degree d with real coefficients, such that $\lim_{n\to\infty} e^{2\pi i P(n)} = 1$. It follows that we also have $\lim_{n\to\infty} e^{2\pi i P(n-1)} = 1$, and consequently

$$\lim_{n\to\infty} e^{2\pi i (P(n)-P(n-1))} = 1$$

But $Q(X) \stackrel{\text{def}}{=} P(X) - P(X-1)$ is a real polynomial of degree $d-1$, and the induction hypothesis implies that $Q(n) \in \mathbb{Z}$ for every $n \in \mathbb{N}$. It follows that,

$$\forall\, n \in \mathbb{N}, \quad P(n) - P(0) = \sum_{k=1}^{n} Q(k) \in \mathbb{Z}. \tag{1}$$

This proves that $e^{2\pi i P(n)} = e^{2\pi i P(0)}$ for every $n \in \mathbb{N}$. Taking the limit as n tend to $+\infty$ we obtain $e^{2\pi i P(0)} = 1$, that is $P(0) \in \mathbb{Z}$. Combining this with (1) we conclude that $P(n) \in \mathbb{Z}$ for every $n \in \mathbb{N}$, and the lemma follows by induction. $\square$

Now, consider a polynomial with real coefficients such that $\lim_{n\to\infty}\{P(n)\} = 0$. (In particular, this is true if $\{P(n)\} < 1/n$ for every $n \in \mathbb{N}$.) Using the well-known inequality $\left|e^{i\theta} - 1\right| \le |\theta|$ we conclude that

$$\left|e^{2\pi i P(n)} - 1\right| = \left|e^{2\pi i\{P(n)\}} - 1\right| \le 2\pi\{P(n)\}$$

and consequently $\lim_{n\to\infty} e^{2\pi i P(n)} = 1$. Applying the Lemma we conclude that $P(n) \in \mathbb{Z}$ for every $n \in \mathbb{N}$, which is the desired conclusion. □

V2-3 Let x_n the sequence defined by $x_n = x_{n-1}^2 - 2$, $n \ge 1$ and $x_0 = 3$. Evaluate

$$\sum_{n\ge 0}\left(\prod_{k=0}^{n} x_k\right)^{-1},$$

if the series converges.

Solution 1: Omran Kouba, *Higher Institute for Applied Sciences and Technology, Damascus, Syria*

Let $\alpha = \ln\left(\frac{3+\sqrt{5}}{2}\right)$ so that $2\cosh(\alpha) = 3 = x_0$. A simple induction argument shows that $x_n = 2\cosh(2^n\alpha)$ for every $n \ge 0$. Now noting that $x_n = \frac{\sinh(2^{n+1}\alpha)}{\sinh(2^n\alpha)}$ we conclude that

$$\left(\prod_{k=0}^{n} x_k\right)^{-1} = \frac{\sinh(\alpha)}{\sinh(2^{n+1}\alpha)}$$

But

$$\frac{\cosh(t)}{\sinh(t)} - \frac{\cosh(2t)}{\sinh(2t)} = \frac{2\cosh^2(t) - \cosh(2t)}{\sinh(2t)} = \frac{1}{\sinh(2t)}.$$

So that,

$$\left(\prod_{k=0}^{n} x_k\right)^{-1} = \sinh(\alpha)\left(\frac{\cosh(2^n\alpha)}{\sinh(2^n\alpha)} - \frac{\cosh(2^{n+1}\alpha)}{\sinh(2^{n+1}\alpha)}\right).$$

Thus, for $m > 1$, we have

$$\sum_{n=0}^{m-1}\left(\prod_{k=0}^{n} x_k\right)^{-1} = \sinh(\alpha)\left(\frac{\cosh(\alpha)}{\sinh(\alpha)} - \frac{\cosh(2^m\alpha)}{\sinh(2^m\alpha)}\right).$$

Taking the limit as m tend to $+\infty$ we see that this series does converge and that

$$\sum_{n=0}^{\infty}\left(\prod_{k=0}^{n} x_k\right)^{-1} = \cosh(\alpha) - \sinh(\alpha) = e^{-\alpha} = \frac{3-\sqrt{5}}{2}.$$

which is the desired conclusion. □

Solution 2: Let $y_n := x^{2^n} + x^{-2^n},\ \ n \geq 0$ where x is any of the roots $\dfrac{3 \pm \sqrt{5}}{2}$ of $x^2 - 3x + 1$. It is immediate to check that y_n satisfies the recurrence relation with the given initial condition, so $y_n = x_n$ and for the given sum equals:

$$\sum_{n\geq 0}\left(\prod_{k=0}^{n} y_k\right)^{-1} = \sum_{n\geq 0}\prod_{k=0}^{n}\frac{x^{2^k}}{(1+x^{2^{k+1}})} = \frac{1}{x}\sum_{n\geq 0}\frac{x^{2^{n+1}}}{(1+x^2)(1+x^4)\cdots(1+x^{2^{n+1}})}$$
$$= \lim_{N\to+\infty}\frac{1}{x}\left(1-\frac{1}{1+x^2}+\sum_{n=0}^{N-1}\frac{x^{2^{n+1}}}{(1+x^2)(1+x^4)\cdots(1+x^{2^{n+1}})}\right)$$
$$= \lim_{N\to+\infty}\frac{1}{x}\left(1-\frac{1}{1+x^2}+\sum_{n=0}^{N-1}\left(\frac{1}{(1+x^2)(1+x^4)\cdots(1+x^{2^n})}-\frac{1}{(1+x^2)(1+x^4)\cdots(1+x^{2^{n+1}})}\right)\right)$$
$$= \frac{1}{x}\left(1-\lim_{N\to+\infty}\frac{1}{(1+x^2)(1+x^4)\cdots(1+x^{2^N})}\right).$$

Now when $x = \dfrac{3+\sqrt{5}}{2} > 1$ the above is clearly equal to $\dfrac{1}{x} = \dfrac{3-\sqrt{5}}{2}$ and, when $(0<)\,x = \dfrac{3-\sqrt{5}}{2} < 1$ is equal to

$$\frac{1}{x}\left(1-\lim_{N\to+\infty}\frac{1}{(1+x^2)(1+x^4)\cdots(1+x^{2^N})}\right) = \frac{1}{x}\left(1-\lim_{N\to+\infty}\frac{1}{\frac{1-x^2}{1-x^2}(1+x^2)(1+x^4)\cdots(1+x^{2^N})}\right)$$
$$= \frac{1}{x}\left(1-\lim_{N\to+\infty}\frac{1-x^2}{1-x^{2^{n+1}}}\right)$$
$$= x.$$

Solution 3: Considering y_n as in the above solution, it easy to see that the given series converges in each case of x, since, when $(0<)\,x = \dfrac{3-\sqrt{5}}{2} < 1$, we have

$$0 < \left(\prod_{k=0}^{n} y_n\right)^{-1} = \frac{x^{2^{n+1}-1}}{(1+x^2)(1+x^4)\cdots(1+x^{2^{n+1}})} < x^{2^{n+1}-1}$$

and when $x = \dfrac{3+\sqrt{5}}{2} > 1$ we have $0 < \left(\displaystyle\prod_{k=0}^{n} y_n\right)^{-1} < \left(\dfrac{1}{x}\right)^{2^{n+1}-1}$.

Since the series converges, we set $\displaystyle\sum_{n\geq 0}\left(\prod_{k=0}^{n} y_k\right)^{-1} = \frac{y_0-\lambda}{2}$ and we will determine λ. A simple inductive argument, using the recurrence relation, shows that

$$\frac{y_n - \lambda y_0 y_1 \cdots y_{n-1}}{2} = \frac{1}{y_n} + \frac{1}{y_n y_{n+1}} + \cdots, \qquad n \geq 1,$$

so $\lambda = \lim\limits_{n\to+\infty} \dfrac{y_n}{y_0y_1\cdots y_n}$.

But from the recurrence relation we have $y_n^2 - 4 = (y_n+2)(y_n-2) = y_{n-1}^2(y_{n-1}^2-4) = y_{n-1}^2y_{n-2}^2(y_{n-2}^2-4) = \cdots (y_0y_1\cdots y_n)^2\,(y_0^2-4)$, so

$$\frac{y_n}{y_0y_1\cdots y_n} = \sqrt{y_0^2 - 4 + \left(\frac{2}{y_0y_1\cdots y_n}\right)^2} \to \sqrt{y_0^2-4+0} = \sqrt{5}$$

which solves the problem. □

Remark: This problem, as well as solutions 2 and 3, appear in [2] with the argument that shows that the series converges in solution 3 added by the editor.

V2-4 *Proposed by Konstantinos Tsouvalas, University of Athens, Athens, Greece.*

Let $a_n = \left(\prod\limits_{k=0}^{n}\binom{n}{k}\right)^{\frac{1}{n(n+1)}}$.

1. Show that $\lim\limits_{n\to+\infty} a_n = \sqrt{e}$ and
2. evaluate $\lim\limits_{n\to+\infty} \dfrac{n(a_n-\sqrt{e})}{\ln n}$, if it exists.

Solution : Omran Kouba, *Higher Institute for Applied Sciences and Technology, Damascus, Syria*

Clearly,

$$\begin{aligned}
\prod_{k=0}^{n}\binom{n}{k} &= \prod_{k=1}^{n}\left(\frac{n(n-1)\cdots(n-k+1)}{k!}\right)\\
&= \prod_{k=1}^{n}\left(\frac{n^2(n-1)^2\cdots(n-k+1)^2}{n!}\cdot\frac{(n-k)!}{k!}\right)\\
&= \prod_{k=1}^{n}\left(\frac{n^2(n-1)^2\cdots(n-k+1)^2}{n!}\right)\cdot\prod_{k=1}^{n}\left(\frac{(n-k)!}{k!}\right)\\
&= \frac{\left(n^n(n-1)^{n-1}\cdots 2^2 1^1\right)^2}{(n!)^{n+1}}
\end{aligned}$$

It follows that

$$\ln(a_n) = \frac{1}{n(n+1)}\sum_{k=1}^{n} 2k\ln k - \frac{1}{n}\ln(n!) \tag{1}$$

Now, let $b_k = \left(k^2 + k + \frac{1}{6}\right)\ln k - \frac{k^2}{2}$. We have

$$\begin{aligned}
b_k - b_{k-1} &= \left(k^2 + k + \frac{1}{6}\right)\ln k - \frac{k^2}{2} - \left(k^2 - k + \frac{1}{6}\right)\ln(k-1) + \frac{(k-1)^2}{2} \\
&= 2k\ln k - k + \frac{1}{2} - \left(k^2 - k + \frac{1}{6}\right)\ln\left(1 - \frac{1}{k}\right) \\
&= 2k\ln k - k + \frac{1}{2} + \left(k^2 - k + \frac{1}{6}\right)\left(\frac{1}{k} + \frac{1}{2k^2} + \frac{1}{3k^3} + O\left(\frac{1}{k^4}\right)\right) \\
&= 2k\ln k + O\left(\frac{1}{k^2}\right)
\end{aligned}$$

This proves that the series $\sum (b_k - b_{k-1} - 2k\ln k)$ is convergent and consequently there is a real constant α such that, for n in the neighborhood of $+\infty$, we have

$$\sum_{k=1}^{n} 2k\ln k = \left(n^2 + n + \frac{1}{6}\right)\ln n - \frac{n^2}{2} + \alpha + o(1)$$

($A = e^{\alpha/2}$ is called the Glaisher-Kinkelin constant.) On the other hand it is well-known $\ln(n!) = \left(n + \frac{1}{2}\right)\ln n - n + \ln\sqrt{2\pi} + O(1/n)$. Replacing in (1), we conclude that for large n we have

$$\ln(a_n) = \frac{1}{2} - \frac{\ln n}{2n} + \frac{1 - \ln(2\pi)}{2n} + O\left(\frac{\ln n}{n^2}\right)$$

Thus

$$a_n = \sqrt{e}\left(1 - \frac{\ln n}{2n} + \frac{1 - \ln(2\pi)}{2n} + O\left(\frac{\ln^2 n}{n^2}\right)\right)$$

In particular,

$$\lim_{n\to\infty} a_n = \sqrt{e} \qquad \text{and} \qquad \lim_{n\to\infty} \frac{n(a_n - \sqrt{e})}{\ln n} = -\frac{\sqrt{e}}{2}.$$

This answers 1. and 2. □

Solution 2: *By the editor*

Euler MacLaurin summation formula[4] says that

> If : $[a, b] \to \mathbb{R}$ where $a, b \in \mathbb{Z}$ is a $2m$-times continuously differentiable function, then for every $m \in \mathbb{N}$
>
> $$\sum_{k=a}^{b} f(k) \;=\; \int_a^b f(x)\,dx + \frac{f(b) + f(a)}{2} + \sum_{k=1}^{m}\left(f^{(2k-1)}(b) - f^{(2k-1)}(a)\right)\frac{B_{2k}}{(2k)!} + R_{m,f,a,b}$$
>
> holds, where
>
> $$R_{m,f,a,b} = -\int_a^b f^{(2m)}(x)\frac{B_{2m}\left(\{x\}\right)}{(2m)!}\,dx$$

[4] see http://en.wikipedia.org/wiki/Euler%E2%80%93Maclaurin_formula

$B_{2m}(\{x\})$ is the 2m-th periodic Bernoulli polynomial[5] and B_{2m} the 2m-th Bernoulli number[6].

Furthermore, for fixed $n \in \mathbb{N}$, the bound $|B_n(\{x\})| \leq \frac{2n!}{(2\pi)^n}\zeta(n)$ for $x \in \mathbb{R}$ is true, where ζ is the Riemann zeta function.[7]

Now for $a = 1,\ b = k,\ m = 1$ and $f(x) = \ln x$ we get, (since $B_2 = 1/6$),

$$\sum_{m=1}^{k} \ln m = \int_1^k \ln x \, dx + \frac{\ln k}{2} + \frac{1}{12}\left(\frac{1}{k} - 1\right) + \frac{1}{2}\int_1^k \frac{B_2\left(\{x\}\right)}{x^2} = k\ln k - k + \frac{\ln k}{2} + \mathcal{O}(1), \qquad k \geq 1, \tag{1}$$

where the constant at the $\mathcal{O}$ is independent of k due to the bound of $B_n(\{x\})$.

Once again, for $a = 1,\ b = n,\ m = 1$ and $f(x) = x\ln x$ we get

$$\begin{aligned}\sum_{k=1}^{n} k\ln k &= \int_1^n x\ln x \, dx + \frac{n\ln n}{2} + \frac{\ln n}{12} + \frac{1}{2}\int_1^n \frac{B_2\left(\{x\}\right)}{x} \\ &= \frac{n^2\ln n}{2} - \frac{n^2}{4} + \frac{n\ln n}{2} + \mathcal{O}\left(\ln n\right), \qquad n \to +\infty,\end{aligned} \tag{2}$$

Now we write

$$\begin{aligned}a_n &= \exp\left(\frac{1}{n(n+1)}\sum_{k=0}^{n}\ln\binom{n}{k}\right) = \exp\left(\frac{1}{n(n+1)}\sum_{k=0}^{n}\left(\ln n! - \ln k! - \ln(n-k)!\right)\right) \\ &= \exp\left(\frac{1}{n}\sum_{k=1}^{n}\ln k - \frac{2}{n(n+1)}\sum_{k=1}^{n}\sum_{m=1}^{k}\ln m\right)\end{aligned}$$

but

$$\begin{aligned}\sum_{k=1}^{n}\sum_{m=1}^{k}\ln m &\overset{(1)}{=} \sum_{k=1}^{n}\left(k\ln k - k + \frac{\ln k}{2} + \mathcal{O}(1)\right) \\ &\overset{(1),(2)}{=\!=\!=} \frac{n^2\ln n}{2} - \frac{3n^2}{4} + n\ln n + \mathcal{O}(n),\end{aligned}$$

so

$$\begin{aligned}a_n &\overset{(1),(2)}{=\!=\!=} \exp\left(\ln n - 1 + \frac{\ln n}{2n} + \mathcal{O}\left(n^{-1}\right) - \frac{2}{n(n+1)}\left(\frac{n^2\ln n}{2} - \frac{3n^2}{4} + n\ln n + \mathcal{O}(n)\right)\right) \\ &= \exp\left(\frac{1}{2} - \frac{\ln n}{2n} + \mathcal{O}\left(n^{-1}\right)\right) \\ &= \sqrt{e} - \frac{\sqrt{e}\ln n}{2n} + \mathcal{O}\left(n^{-1}\right)\end{aligned}$$

[5] see `http://en.wikipedia.org/wiki/Bernoulli_polynomial`

[6] see `http://en.wikipedia.org/wiki/Bernoulli_numbers`

[7] see `http://en.wikipedia.org/wiki/Euler%E2%80%93Maclaurin_formula#The_remainder_term`

which answers both parts of the problem. $\square$

Solution 3 (for part 1): *By the editor*
The following elementary Lemma can help us avoid the use of Euler MacLaurin summation formula, however, the estimate that can be made based on this for the sums $\sum_{m=1}^{k} \ln m$ and $\sum_{k=1}^{n} k \ln k$ is not that accurate to answer the second part of the problem.

Lemma. *Let* $f : [M, N] \to \mathbb{R}$ *be a monotone function. Then*

$$\left| \sum_{k=M}^{N} f(k) - \int_M^N f(x)\, dx \right| \le \max_{x \in [M,N]} \{|f(M)|, |f(N)|\}.$$

Proof. Assume that f is decreasing and the other case we can work with $-f$. We clearly have $f(k+1) \le \int_k^{k+1} f(x)\, dx \le f(k)$ for k=M,. . . ,N-1 so summing for these values of k we get

$$-f(M) + \sum_{k=M}^{N} f(k) \le \int_M^N f(x)\, dx \le \sum_{k=M}^{N} f(k) - f(N),$$

which gives the desired result. $\square$

Applying the Lemma on $f(x) = \ln x$ on $[1, k]$ we get $\left| \int_1^k \ln x\, dx - \sum_{m=1}^{k} \ln m \right| \le \ln k, \quad k \ge 1$, so

$$\sum_{m=1}^{k} \ln m = \int_1^k \ln x\, dx + \mathcal{O}(\ln k) = k \ln k - k + \mathcal{O}(\max\{1, \ln k\}), \qquad k \ge 1.$$

Applying it again on $f(x) = x \ln x$ on $[1, n]$ we get

$$\sum_{k=1}^{n} k \ln k = \int_1^n x \ln x\, dx + \mathcal{O}(n \ln n) = \frac{n^2 \ln n}{2} - \frac{n^2}{4} + \mathcal{O}(n \ln n), \qquad n \to +\infty.$$

Now plugging the above on $a_n = \exp\left(\frac{1}{n} \sum_{k=1}^{n} \ln k - \frac{2}{n(n+1)} \sum_{k=1}^{n} \sum_{m=1}^{k} \ln m \right)$ we get

$$a_n = \sqrt{e} + \mathcal{O}\left(\frac{\ln n}{n} \right).$$

$\square$

V2-5 Evaluate $\displaystyle\int_0^1 \sqrt{4x-4x^2}\tanh^{-1}\left(\sqrt{4x-4x^2}\right)\,dx.$

Solution 1: Omran Kouba, *Higher Institute for Applied Sciences and Technology, Damascus, Syria*

Let us denote the considered integral by I. The graph of the integrand is symmetric with respect to the line $x = 1/2$ so

$$I = 2\int_{1/2}^1 \sqrt{4x-4x^2}\tanh^{-1}\left(\sqrt{4x-4x^2}\right)\,dx$$

Next we make the change of variables $t = \dfrac{2x-1}{1+\sqrt{4x-4x^2}} \iff x = \dfrac{(1+t)^2}{2(1+t^2)}$, we find that

$$I = -\int_0^1 \frac{2(1-t^2)^2}{(1+t^2)^3}\ln t\,dt$$

Integrating by parts, after noting that

$$\left(\frac{x-x^3}{(1+x^2)^2} + \arctan x\right)' = \frac{2(1-x^2)^2}{(1+x^2)^3}$$

we conclude that

$$\begin{aligned}
I &= -\left[\left(\frac{t-t^3}{(1+t^2)^2} + \arctan t\right)\ln t\right]_0^1 + \int_0^1\left(\frac{1-t^2}{(1+t^2)^2} + \frac{1}{t}\arctan t\right)\,dt \\
&= \int_0^1 \frac{1-t^2}{(1+t^2)^2}\,dt + \int_0^1 \frac{\arctan t}{t}\,dt \\
&= \left[\frac{t}{1+t^2}\right]_0^1 + \int_0^1 \frac{\arctan t}{t}\,dt \\
&= \frac{1}{2} + \int_0^1 \frac{\arctan t}{t}\,dt
\end{aligned}$$

The remaining integral equals the well-known Catalan constant G

$$G = \int_0^1 \frac{\arctan t}{t}\,dt = \int_0^1 \frac{-\ln t}{1+t^2}\,dt = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2} \approx 0.915965594177.$$

So, $I = \dfrac{1}{2} + G \approx 1.415965594177.$ $\square$

Solution 2: *By the editor*

We have:

$$\begin{aligned}
S :&= \int_0^1 \sqrt{4x(1-x)}\tanh^{-1}\sqrt{4x(1-x)}\,dx \\
&= \frac{1}{2}\int_0^1 \sqrt{4x(1-x)}\ln\left(\frac{1+\sqrt{4x(1-x)}}{1-\sqrt{4x(1-x)}}\right)\,dx && (x=\cos^2 y)\\
&= \frac{1}{2}\int_0^{\pi/2} \sin^2(2y)\ln\left(\frac{1+\sin(2y)}{1-\sin(2y)}\right)\,dy && (2y=\pi/2-x)\\
&= \frac{1}{4}\int_{-\pi/2}^{\pi/2} \cos^2 x\ln\left(\frac{1+\cos x}{1-\cos x}\right)\,dx \\
&= \frac{1}{2}\int_0^{\pi/2} \cos^2 x\ln\left(\frac{1+\cos x}{1-\cos x}\right)\,dx && \left(\frac{1+\cos x}{1-\cos x}=\cot^2\frac{x}{2}\right)\\
&= \int_0^{\pi/2} \cos^2 x\ln\left(\cot\frac{x}{2}\right)\,dx && (x=2u)\\
&= 2\int_0^{\pi/4} \cos^2(2u)\ln(\cot u)\,du =: 2I.
\end{aligned}$$

Setting $J := \int_0^{\pi/4} \sin^2(2u)\ln(\cot u)\,du$ we directly see integrating by parts that $I - J = \int_0^{\pi/4} \cos(4u)\ln(\cot u)\,du = \frac{1}{2}$, so

$$J = I - \frac{1}{2} \tag{1}$$

But also $I = \int_0^{\pi/4}(1-\sin^2(2u))\ln(\cot u)\,du = G - J \overset{(1)}{=} G + \frac{1}{2} - I$, where $G = \int_0^{\pi/4} \ln(\cot x)\,dx$ is the Catalan's constant,[8] so $S = 2I = G + \frac{1}{2}$. □

Remark: Using the identity $\binom{n}{k}^{-1} = (n+1)\int_0^1 t^k(1-t)^{n-k}$, the given integral can be transformed to the sum $\sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}(2n+1)(2n-1)}$ which has been discussed on the Art Of Problem Solving forum (`http://www.artofproblemsolving.com/Forum/viewtopic.php?f=296&t=511138`) and is the source of the problem.

V2-**6** *Proposed by the editor*

Let k be a positive integer. Show that

$$\sum_{n\geq 1} \frac{1}{(2n-1)(2n-3)\cdots(2n-2k-1)} = \frac{(-1)^k 2^{k-1}}{k\cdot k!\binom{2k}{k}}.$$

[8]see `http://en.wikipedia.org/wiki/Catalan%27s_constant#Integral_identities`

Solution : Omran Kouba, *Higher Institute for Applied Sciences and Technology, Damascus, Syria*

For integers n and k such that $k \geq 0$, we define $A_n^{(k)}$ by

$$A_n^{(k)} = \frac{2^k\, k!}{(2n-1)(2n-3)\cdots(2n-2k-1)}$$

Clearly, for $k > 0$ we have

$$\begin{aligned} A_n^{(k-1)} - A_{n+1}^{(k-1)} &= \frac{2^{k-1}\,(k-1)!}{(2n-1)(2n-3)\cdots(2n-2k+1)} - \frac{2^{k-1}\,(k-1)!}{(2n+1)(2n-1)\cdots(2n-2k+3)} \\ &= \frac{2^{k-1}\,(k-1)!(2n+1-(2n-2k+1))}{(2n+1)(2n-1)\cdots(2n-2k+1)} \\ &= \frac{2^k\, k!}{(2n+1)(2n-1)\cdots(2n-2k+1)} = A_{n+1}^k \end{aligned}$$

Taking the sum as n varies from 0 to $m-1$ we obtain

$$\sum_{n=1}^{m} A_n^{(k)} = \sum_{n=0}^{m-1} A_{n+1}^{(k)} = \sum_{n=0}^{m-1} (A_n^{(k-1)} - A_{n+1}^{(k-1)}) = A_0^{(k-1)} - A_m^{(k-1)}$$

Letting m tend to $+\infty$ we obtain

$$\sum_{n=1}^{\infty} A_n^{(k)} = A_0^{(k-1)} = \frac{(-1)^k 2^{k-1}\,(k-1)!}{1\cdot 3\cdots(2k-1)} = \frac{(-1)^k 2^{2k-1}\,(k-1)!\,k!}{(2k)!}.$$

Dividing both sides by $2^k\,k!$ we obtain the desired result. □

Remark *by the editor:*

The solution I had given to this one is so lengthy that I wouldn't even dare to write. The only "advantage" I can see over the beautiful telescoping trick of the above solution is that it may lead to a solution in cases that we cannot telescope the sum and that it may give (non trivial) identities as side results. For example one can show using this method that for $\mathbb{N} \ni k \geq 3$:

$$\sum_{n\geq 0} \frac{1}{(2n+1)(3n+2)\cdots(kn+k-1)} =$$

$$\frac{1}{k!}\sum_{m=2}^{k} (-1)^{m-1}\binom{k}{m}(m-1)m^{k-2}\left(\frac{\pi}{2}\cot\frac{\pi}{m} - \ln m + \sum_{\ell=1}^{m-1}\cos\frac{2\ell\pi}{m}\cdot\ln\left(\sin\frac{\ell\pi}{m}\right)\right).$$

The main steps are:

1. decompose the summand to partial fractions
2. changing the order of summation after writing $\sum_{n\geq 1} = \lim_{N\to+\infty}\sum_{n=1}^{N}$,
3. writing the inner sum in terms of the Digamma function $\Psi(x)$ (`http://en.wikipedia.org/wiki/Digamma_function`)

4. use that $\Psi(x) = \ln x + \mathcal{O}(x^{-1}) \qquad x \to +\infty$ and maybe some other properties of this function.[9]

V2-7 *Proposed by the editor*

Evaluate $\displaystyle\sum_{k=0}^{n} (-1)^{k+1} \frac{\binom{n}{k}}{2k-1}$.

Solution 1: Omran Kouba, *Higher Institute for Applied Sciences and Technology, Damascus, Syria*

Consider the rational fraction

$$R_n(X) = \frac{n!}{X(X+1)\cdots(X+n)}$$

We know that there exist $\lambda_0, \lambda_1, \ldots, \lambda_n$ such that

$$R_n(X) = \sum_{j=0}^{n} \frac{\lambda_k}{X+k}$$

where λ_k can be calculated by

$$\lambda_k = \lim_{x \to -k} (x+k) R_n(x) = \frac{n!}{(-k)(1-k)\cdots(-1)(1)(2)\cdots(n-k)} = (-1)^k \binom{n}{k}$$

Thus

$$R_n(X) = \sum_{k=0}^{n} (-1)^k \frac{\binom{n}{k}}{X+k}$$

In particular, for $X = -1/2$ we obtain

$$\sum_{k=0}^{n} (-1)^{k+1} \frac{\binom{n}{k}}{2k-1} = \frac{2^n n!}{1 \cdot 3 \cdots (2n-1)} = \frac{4^n}{\binom{2n}{n}}$$

□

Solution 2: *Moubinool Omarjee, Lycée Henri IV, Paris, France*

Denote $A(n) := \displaystyle\sum_{k=0}^{n} (-1)^{k+1} \frac{\binom{n}{k}}{2k-1}$. Then

$$\begin{aligned}
A(n) &= 1 - \sum_{k=1}^{n} (-1)^k \frac{\binom{n}{k}}{2k-1} = 1 - \sum_{k=1}^{n} \binom{n}{k} (-1)^k \int_0^1 t^{2k-2}\, dt \\
&1 - \int_0^1 \frac{\sum_{k=1}^{n} \binom{n}{k} (-1)^k t^{2k}}{t^2}\, dt = \\
&= 1 - \int_0^1 \frac{\sum_{k=0}^{n} \binom{n}{k} (-1)^k t^{2k} - 1}{t^2}\, dt = \\
&= 1 - \int_0^1 \frac{(1-t^2)^n - 1}{t^2}\, dt.
\end{aligned}$$

[9] see `http://www.frm.utn.edu.ar/analisisdsys/MATERIAL/Funcion_Gamma.pdf` for example.

Denoting $B(n) := \int_0^1 \frac{(1-t^2)^n - 1}{t^2}\,dt$, an integration by parts gives

$$B(n) = \frac{(1-t^2)^n - 1}{-t}\Bigg|_0^1 - \int_0^1 \frac{n(-2t)(1-t^2)^{n-1}}{-t}\,dt = 1-2n\int_0^1 (1-t^2)^{n-1}\,dt \overset{t=\cos u}{=} 1-2n\int_0^{\pi/2} \sin^{2n-1} u\,du$$

So $A(n)$ can be written in terms of Wallis' integral[10] and we get

$$\begin{aligned} A(n) &= 2n\int_0^1 \sin^{2n-1} u\,du = 2n\frac{(2n-2)(2n-4)\cdots 2}{(2n-1)(2n-3)\cdots 3\cdot 1} \\ &= \frac{((2n)(2n-2)(2n-4)\cdots 2)^2}{((2n)(2n-1)(2n-2)\cdots 3\cdot 2\cdot 1)} = \frac{(2^n n!)^2}{(2n)!} = \frac{4^n}{\binom{2n}{n}}. \end{aligned}$$

□

Comment by the editor: The above solution holds for $n \geq 1$. But trivially $A(0) = \frac{4^0}{\binom{2\cdot 0}{0}}$ so $A(n) = \frac{4^n}{\binom{2n}{n}}$ for $n \geq 0$.

Solution 3: *by the editor*

This one came up from the solution I gave to problem V2-**6**. Using the method Omran Kouba used on Solution 1 to determine the partial fractions decomposition of $R_n(X)$, we can see that

$$\frac{1}{(2n-1)(2n-3)\cdots(2n-k-1)} = \sum_{m=0}^{k} \frac{(-1)^{k-m}\binom{k}{m}}{2^k k!}\cdot\frac{1}{2n-2m-1}$$

so, for $n = 1$ on the above we get the desired result. □

V2-8 Let $A_{n,m,k} := \frac{(-1)^{m-1}}{m2^n n^m}\binom{n}{k}k^m$, where m, n are positive integers and k is a non-negative integer.

1. Can we find a sequence $\{a_m\}_{m\geq 1}$ and $n_0 \in \mathbb{N}$ such that $\mathbb{N} \ni n \geq n_0 \Rightarrow \left|\sum_{k=0}^{n} A_{n,m,k}\right| < a_m$ for every m, with $\sum_{m=1}^{+\infty} a_m$ being convergent?
2. (*) Is it true that, in the case that $\lim_{n\to+\infty}\sum_{k=0}^{n} A_{n,m,k} = a_m \in \mathbb{R}\quad m \geq 1$ with $\sum_{m\geq 1} a_m$ convergent, then $\lim_{n\to+\infty}\sum_{k=0}^{n}\sum_{m\geq 1} A_{n,m,k} = \sum_{m\geq 1} a_m$?

[10] see `http://en.wikipedia.org/wiki/Wallis%27_integrals`

3. Evaluate

$$\lim_{n\to+\infty}\frac{1}{n}\left(\prod_{\ell=0}^{n}(n+\ell)^{C_n^\ell}\right)^{\frac{1}{2^n}},\qquad \text{where } C_n^\ell=\binom{n}{\ell},$$

if it exists.

Solution 1: Omran Kouba, *Higher Institute for Applied Sciences and Technology, Damascus, Syria*

1. Note that for every n and m we have

$$\left|\sum_{k=0}^{n}A_{n,m,k}\right|=\frac{1}{m2^n n^m}\sum_{k=0}^{n}\binom{n}{k}k^m\geq\frac{1}{m2^n n^m}\binom{n}{n}n^m=\frac{2^{-n}}{m}$$

Thus, for every $n_0\in\mathbb{N}$ we have

$$\sup_{n\geq n_0}\left|\sum_{k=0}^{n}A_{n,m,k}\right|\geq\frac{2^{-n_0}}{m}$$

and consequently

$$\sum_{m\geq1}\left(\sup_{n\geq n_0}\left|\sum_{k=0}^{n}A_{n,m,k}\right|\right)=+\infty$$

This answers negatively the first question.

2. We will use Bernstein's proof of Weirstrass' theorem.

Theorem (Weirstrass). *Let* $f:[0,1]\longrightarrow\mathbb{R}$ *be a continuous function. Then the sequence of functions* $\{B_n(f,\cdot)\}_{n\geq0}$ *defined by*

$$B_n(f,x)=\sum_{k=0}^{n}\binom{n}{k}f\left(\frac{k}{n}\right)x^k(1-x)^{n-k}$$

converges uniformly on $[0,1]$ *to* f.

Applying this to $f_m(x)=\dfrac{(-x)^m}{m}$ and noting that $\sum_{k=0}^{n}A_{n,m,k}=B_n\left(f_m,\dfrac{1}{2}\right)$ we conclude that

$$\lim_{n\to\infty}\sum_{k=0}^{n}A_{n,m,k}=f_m\left(\frac{1}{2}\right)=\frac{(-1)^{m-1}}{m2^m}\stackrel{\text{def}}{=}a_m$$

Using the well-known expansion $\ln(1+x)=\sum_{m=1}^{\infty}\dfrac{(-1)^{m-1}}{m}x^m$ that is valid for $x\in(-1,1]$, we conclude that

$$\sum_{m\geq1}a_m=\ln\left(1+\frac{1}{2}\right)=\ln\left(\frac{3}{2}\right)\tag{1}$$

On the other hand, using the same expansion we have

$$\sum_{m\geq1}A_{n,m,k}=\frac{1}{2^n}\binom{n}{k}\sum_{m\geq1}\frac{(-1)^{m-1}}{m}\left(\frac{k}{n}\right)^m=\frac{1}{2^n}\binom{n}{k}\ln\left(1+\frac{k}{n}\right)$$

It follows that

$$\sum_{k=0}^{n}\left(\sum_{m\geq 1}A_{n,m,k}\right)=B_n\left(f,\frac{1}{2}\right)$$

where $f:[0,1]\to\mathbb{R}, f(x)=\ln(1+x)$. Thus, applying the same theorem again we conclude that

$$\lim_{n\to\infty}\sum_{k=0}^{n}\left(\sum_{m\geq 1}A_{n,m,k}\right)=\lim_{n\to\infty}B_n\left(f,\frac{1}{2}\right)=f\left(\frac{1}{2}\right)=\ln\left(\frac{3}{2}\right) \quad (2)$$

Comparing (1) and (2) we see that

$$\lim_{n\to\infty}\sum_{k=0}^{n}\left(\sum_{m\geq 1}A_{n,m,k}\right)=\sum_{m\geq 1}a_m.$$

This answers 2 positively.

3. We have shown in (2) that

$$\lim_{n\to\infty}\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\ln\left(1+\frac{k}{n}\right)=\ln\left(\frac{3}{2}\right)$$

Using the continuity of $x\mapsto e^x$ we conclude that

$$\lim_{n\to\infty}\frac{1}{n}\left(\prod_{k=0}^{n}(n+k)^{\binom{n}{k}}\right)^{2^{-n}}=\frac{3}{2},$$

which is the desired conclusion. $\square$

Solution 2 (for part 3): *Michael Lambrou, University of Crete, Heraklion, Crete, Greece.*

Setting $A(n):=\frac{1}{2^n}\sum_{k=0}^{n}\ln\left(1+\frac{k}{n}\right)^{C_n^k}$, we show that $A_n\to\ln(3/2)$. We use that

1. $\binom{n}{k}=\binom{n}{n-k}$,
2. $$\sum_{k=0}^{n}\binom{n}{k}=2^n\wedge\sum_{k=0}^{n}\binom{n}{k}k=2^{n-1}n\wedge\sum_{k=0}^{n}\binom{n}{k}k^2=2^{n-2}(n+n^2)$$
$$\Rightarrow\sum_{k=0}^{n}\binom{n}{k}(n^2-4kn+4k^2)=2^n n,$$
3. For $y>x>0$, from the Mean Value Theorem in $[x,y]$ we have that $\log y-\log x<\frac{1}{x}(y-x)$.

From 3. we have

$$0 \le \log\frac{9}{4} - \log\left(2 + \frac{k(n-k)}{n^2}\right) \le \frac{1}{2 + \frac{k(n-k)}{n^2}}\left(\frac{9}{4} - 2 - \frac{k(n-k)}{n^2}\right)$$
$$\le \frac{1}{2}\left(\frac{9}{4} - 2 - \frac{k(n-k)}{n^2}\right) = \frac{n^2 - 4kn + 4k^2}{8n^2}.$$

Writing the sum backwards, from 1.:

$$\sum_{k=0}^{n}\binom{n}{k}\log(1+\frac{k}{n}) = \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\log(1+\frac{k}{n}) + \frac{1}{2}\sum_{k=0}^{n}\binom{n}{n-k}\log(1+\frac{n-k}{n})$$
$$= \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\left(\log(1+\frac{k}{n}) + \log(1+\frac{n-k}{n})\right)$$
$$= \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\log\left(2 + \frac{k(n-k)}{n^2}\right),$$

hence

$$\left|\log(3/2) - \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\log\left(1+\frac{k}{n}\right)\right| = \left|\frac{1}{2}\log\frac{9}{4} - \frac{1}{2}\cdot\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\log\left(2+\frac{k(n-k)}{n^2}\right)\right|$$
$$= \left|\frac{1}{2}\cdot\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\left(\log\frac{9}{4} - \log\left(2+\frac{k(n-k)}{n^2}\right)\right)\right|$$
$$\le \left|\frac{1}{2}\cdot\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\frac{n^2-4kn+4k^2}{8n^2}\right|$$
$$\overset{2.}{=} \frac{1}{2}\cdot\frac{1}{2^n}\cdot\frac{2^n n}{8n^2} = \frac{1}{16n} \to 0.$$

□

Remarks: 1) Part 3 of the above problem has been discussed on different places around the web, such as the Art Of Problem Solving forum and the Romanian Mate Forum. It has also been discussed on the Greek forum `www.mathematica.gr` (see `http://www.mathematica.gr/forum/viewtopic.php?f=9&t=10632`) where the above solution was given, an other approach by Demetres Christofides, and a partial solution by the editor, where the justification of part 2. here was missing.

2) Demetres Christofides also answered part 1 negatively.

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