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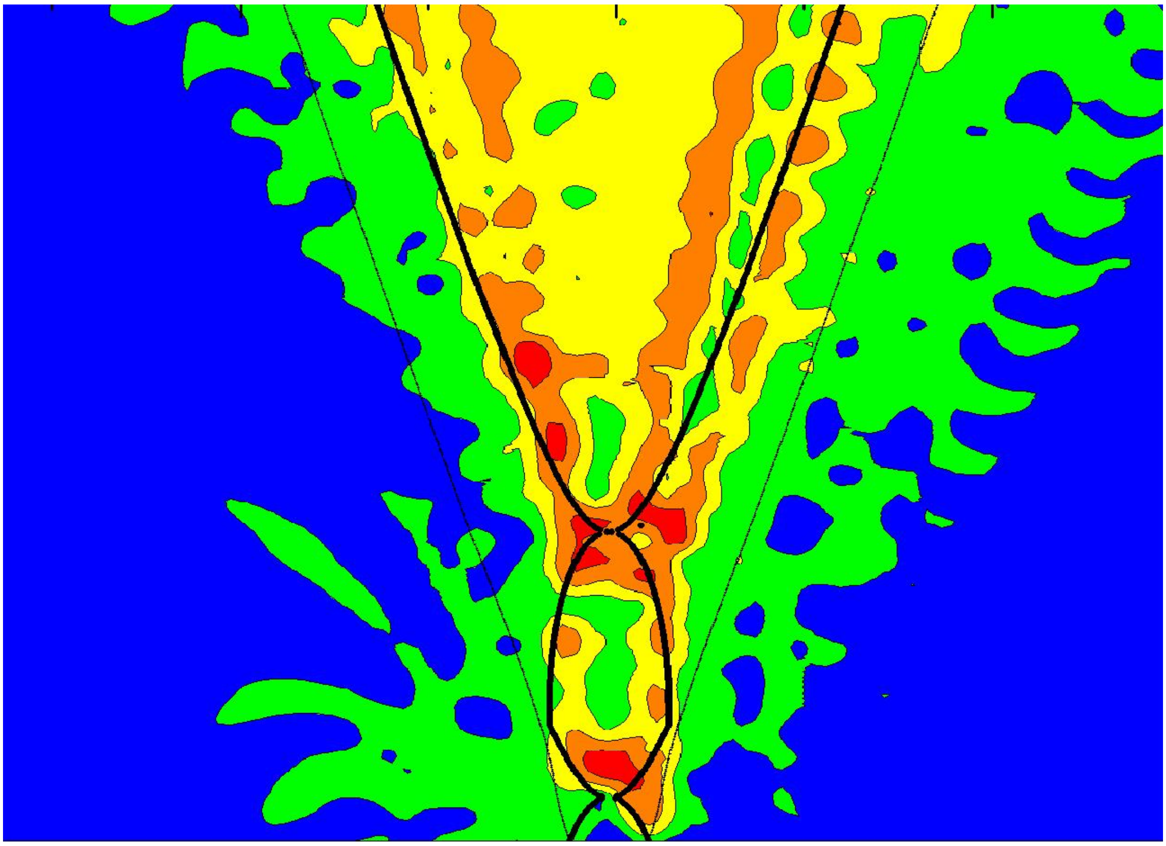

Picture by M.Miliaresi's graduate thesis "Dynamics of asteroids in resonance with Jupiter".

Editor

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Asymmetry is an electronic mathematical journal which aims at giving the chance to the readers, and the editor himself, to work on interesting mathematical problems or find information about various mathematical topics. The problems presented may either be original or taken from the existing literature or the web. Attempt will be made to be precise as regards the problems' original source. The level of the topics is undergraduate and beyond. Readers are encouraged by the editor to submit proposals and/or solutions to proposed problems. Proposals and solutions are preferred to be in LaTeX format using what is necessary from the preamble presented in `http://www.asymmetry.gr/index.php?option=com_content&view=article&id=3`, must be legible and should appear on separate sheets, each indicating the name of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that no solution is available at the time the problem is published. Questions concerning proposals and/or solutions can be sent by e-mail to akotronis@gmail.com.

Editor's note

The editor would like to invite the interested reader to visit the new mathematical forum

www.mathimatikoi.org

This forum is for university level mathematics, in the English language, and was made by a group of Greek mathematicians.

Problems and Solutions

The source of the problems will appear along with the publication of the solutions

The Problems

V4-1 *Proposed by the editor*

Show that $\displaystyle\sum_{n\geq 1}\frac{(-1)^{n-1}}{n(n+1)\cdots(n+k)}=\frac{2^k}{k!}\left(\ln 2-\sum_{i=1}^{k}\frac{(1/2)^i}{i}\right)$

where k is a non negative integer and for $k=0$ the second sum is considered to be 0.

V4-**2** *Proposed by the editor*

Show that $\displaystyle\sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-2k)^{n+2}=\frac{2^n n(n+2)!}{6}$.

V4-**3** *Proposed by the editor*

Let n be a non negative integer, m a positive integers and $x\in\mathbb{C}$. Show that for the values of n, m, x for which the denominators don't vanish, the following identity holds:

$$\sum_{k=0}^{n}(-1)^k\frac{\binom{n}{k}\binom{x}{m-k}}{(m+n-k)\binom{x+n}{m+n-k}}=\frac{1}{m}\delta_{n0},$$

where $\delta_{n0}=\begin{cases}1 & ,n=0\\ 0 & ,n\neq 0\end{cases}$ is Kronecker's delta.

V4-**4** *Proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France*

For $n\in\mathbb{Z}$, let

$$a_n:=\int_0^1\int_0^1 e^{-|x-y|+2n\pi(x-y)i}\,dx\,dy,\qquad I_N:=\{(m,n)\in\mathbb{Z}^2: |m|\geq N,\ \text{or } |n|\geq \mathbb{N}\}\qquad\text{and}$$

$$S_N:=\sum_{(m,n)\in I_N}a_m a_n.$$

Evaluate $\displaystyle\lim_{N\to+\infty} NS_N$, if it exists.

V4-**5** *Proposed by Serafeim Tsipelis, Ioannina, Greece.*

Show that

$$\int_0^{\pi/2}x\log(1-\cos x)\,dx=\frac{35}{16}\zeta(3)-\frac{\pi^2\log 2}{8}-\pi G,$$

where G is the Catalan's constant[1] and ζ is the Riemann's zeta function.[2]

V4-**6** *Proposed by Serafeim Tsipelis, Ioannina, Greece.*

Evaluate $\displaystyle\int_0^{+\infty}\frac{\ln(\cos^2 x)}{1+e^{2x}}\,dx$.

V4-**7** (*) *Proposed by Cody Thompson, Chaneysville, Pennsylvania, USA, Allegany College of Maryland.*

Show that $\displaystyle\int_0^1\frac{\ln(x)\ln(1-x)\ln^2(1+x)}{x}dx=\frac{7\pi^2\zeta(3)}{48}-\frac{25}{16}\zeta(5)$.

[1] `http://mathworld.wolfram.com/CatalansConstant.html`

[2] `http://en.wikipedia.org/wiki/Riemann_zeta_function`.

V4-**8** (*) *Proposed by Konstantinos Tsouvalas, University of Athens, Athens, Greece.*

1. Show that $\lim\limits_{n\to+\infty}\left(\frac{2}{3}\right)^n\sum\limits_{k=0}^{[n/3]}\binom{n}{k}2^{-k}=\frac{1}{2}$ without using probabilistic methods.

2. Can we find a better approximation of the quantity $\left(\frac{2}{3}\right)^n\sum\limits_{k=0}^{[n/3]}\binom{n}{k}2^{-k}$ than
$$\left(\frac{2}{3}\right)^n\sum_{k=0}^{[n/3]}\binom{n}{k}2^{-k}=\frac{1}{2}+o(1)\quad ?$$

$[\cdot]$ denotes the integer part.

Solutions

V3-1 *Proposed by José Luis Díaz–Barrero, Technical University of Catalonia (BARCELONA TECH), Barcelona, Spain.*

Let $a_1, a_2, \ldots, a_n$, be $n \geq 2$ positive real numbers. Prove that
$$\frac{a_1}{a_2+3\sqrt[7]{a_1a_2^6}}+\frac{a_2}{a_3+3\sqrt[7]{a_2a_3^6}}+\ldots+\frac{a_n}{a_1+3\sqrt[7]{a_na_1^6}}\geq\frac{n}{4}.$$

Solution 1: *Ángel Plaza, Department of Mathematics, Universidad de Las Palmas de Gran Canaria, Spain*

By the AM-GM inequality $\sqrt[7]{a_1a_2^6}\leq\frac{a_1+6a_2}{7}$ and then
$$\frac{a_1}{a_2+3\sqrt[7]{a_1a_2^6}}\geq\frac{a_1}{\frac{3}{7}a_1+\frac{25}{7}a_2},$$

and similar inequalities hold for the other quotients.

We may consider new variables $x_1=\frac{a_2}{a_1}$, $x_2=\frac{a_3}{a_2}$, $x_n=\frac{a_1}{a_n}$. Note that the new variables are positive and their product is equal 1. The inequality reads now as
$$\sum_{\text{cyclic}}\frac{1}{\frac{3}{7}+\frac{25}{7}x_n}\geq\frac{n}{4}.$$

Since function $f(x)=\frac{1}{\frac{3}{7}+\frac{25}{7}x}$ is decreasing in $(0,1]$, then $\min\limits_{x\in(0,1]}f(x)=f(1)=\frac{1}{4}$ and the desired inequality follows. $\square$

Solution 2: *Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, 00133 Roma, Italy*

The inequality is equivalent to

$$\frac{1}{\frac{a_2}{a_1}+3\sqrt[7]{\frac{a_2^6}{a_1^6}}}+\frac{1}{\frac{a_3}{a_2}+3\sqrt[7]{\frac{a_3^6}{a_2^6}}}+\ldots+\frac{1}{\frac{a_1}{a_n}+3\sqrt[7]{\frac{a_1^6}{a_n^6}}}\geq\frac{n}{4}$$

Now we define

$$\frac{a_1}{a_n}=x_1,\ \frac{a_2}{a_1}=x_2,\ldots,\frac{a_n}{a_{n-1}}=x_n$$

and then

$$\prod_{k=1}^{n}x_k=1\implies\sum_{cyc}\frac{1}{x_k+3\sqrt[7]{x_k^6}}\geq\frac{n}{4}$$

By $x_k=e^{t_k}$ we get

$$\sum_{k=1}^{n}t_k=0\implies\sum_{cyc}\frac{1}{e^{t_k}+3e^{6t_k/7}}\geq\frac{n}{4}$$

The function $f(x)=e^x+3e^{6x/7}$ verifies

$$f''=\frac{1}{49}\frac{e^t(49e^t+249e^{\frac{6t}{7}}+324e^{\frac{5t}{7}})}{(e^t+3e^{\frac{6t}{7}})^3}>0$$

thus convex. We can apply Karamata's theorem:

Let $t_1\geq t_2\geq t_3\geq\ldots\geq t_n$ and $\sum_{k=1}^{n}t_k=0$ (we can order the variables since the inequality is symmetric). Since

$$\sum_{k=1}^{j}t_k\geq 0,\quad\forall\, 1\leq j<n,$$

the theorem yields

$$\sum_{k=1}^{n}f(t_k)\geq\sum_{k=1}^{n}f(0)=\frac{n}{4}$$

$\square$

Solution 3: *By the proposer*

Consider the functionf : $(0,+\infty)\to\mathbb{R}$ defined by $f(t)=\dfrac{t^7}{1+3t}$. Since $f'(t)>0$ and $f''(t)>0$, as we will see later on, then f is increasing and convex. Setting $t_k=\sqrt[7]{\dfrac{a_k}{a_{k+1}}}$, $(1\leq k\leq n-1)$, and $t_n=\sqrt[7]{\dfrac{a_n}{a_1}}$ into $f(t)$, we get

$$\frac{1}{n}\sum_{k=1}^{n}f(t_k)\geq f\left(\frac{1}{n}\sum_{k=1}^{n}t_k\right)\geq f\left(\left[\prod_{k=1}^{n}t_k\right]^{1/n}\right)=f(1)=\frac{1}{4}$$

on account of Jensen's and AM-GM inequalities. Using the preceding inequalities, we have

$$\frac{1}{n}\sum_{k=1}^{n}\frac{a_k/a_{k+1}}{1+3\sqrt[7]{a_k/a_{k+1}}}\geq\frac{1}{4},$$

where $a_{n+1}=a_1$ and the statement immediately follows. Equality holds when $a_1=a_2=\ldots=a_n$.

Finally, we will prove that f is increasing and convex. Indeed, since

$$f'(t)=\frac{7t^6}{1+3t}-\frac{3t^7}{(1+3t)^2}=\frac{t^6(7+18t)}{(1+3t)^2}$$

and

$$f''(t)=\frac{6t^5(45t^2+35t+7)}{(1+3t)^3}$$

then, for all $t>0$, trivially holds that $f'(t)>0$ and $f''(t)>0$. This completes the proof. □

Solution 4: Omran Kouba, *Higher Institute for Applied Sciences and Technology, Damascus, Syria*

We will use the following property :

Lemma. *Let λ and α be two real numbers such that $\lambda>0$ and $\alpha\in[3-2\sqrt{2},3+2\sqrt{2}]$, and let φ be the function defined on $\mathbb{R}$ by $\varphi(x)=\dfrac{1}{e^x+\lambda e^{\alpha x}}$. Then φ is convex.*

Proof. Indeed, if $\psi(x)=e^x+\lambda e^{\alpha x}$ then $\varphi''\geq 0$ if and only if $2\psi'^2-\psi\psi''\geq 0$, because ψ is positive. But

$$\begin{aligned}2\psi'^2(x)-\psi(x)\psi''(x)&=e^{2x}-\lambda(1-4\alpha+\alpha^2)e^{(1+\alpha)x}+\lambda^2\alpha^2e^{2\alpha x}\\&=\lambda\alpha e^{(1+\alpha)x}\left(\frac{e^{(1-\alpha)x}}{\lambda\alpha}+\frac{\lambda\alpha}{e^{(1-\alpha)x}}+4-\alpha-\frac{1}{\alpha}\right)\\&\geq\lambda\alpha e^{(1+\alpha)x}\left(6-\alpha-\frac{1}{\alpha}\right)\geq 0\end{aligned}$$

where the last inequality follows from the fact that $6-\alpha-\alpha^{-1}\geq 0$ if and only if α belongs to the interval $[3-2\sqrt{2},3+2\sqrt{2}]$. □

Corollary. *Let λ and α be two real numbers such that $\lambda>0$ and $\alpha\in[3-2\sqrt{2},3+2\sqrt{2}]$. Then, for every n (≥ 2) positive real numbers $t_1,t_2,\ldots,t_n$ such that $t_1t_2\cdots t_n=1$ we have*

$$\sum_{k=1}^{n}\frac{1}{t_k+\lambda t_k^{\alpha}}\geq\frac{n}{1+\lambda}$$

Proof. Indeed this is just Jensen's inequality: $\displaystyle\frac{1}{n}\sum_{k=1}^{n}\varphi(x_k)\geq\varphi\left(\frac{1}{n}\sum_{k=1}^{n}x_k\right)$, with $x_k=\ln t_k$, for $k=1,2,\ldots,n$. □

Now consider n positive numbers $a_1, a_2, \ldots, a_n$, and apply the corollary to $t_1, \ldots, t_n$ with $t_k = a_{k+1}/a_k$, for $k = 1, \ldots, n$, ($a_{n+1} = a_1$.) It follows that

$$\sum_{k=1}^{n} \frac{a_k}{a_{k+1} + \lambda a_k^{1-\alpha} a_{k+1}^{\alpha}} \geq \frac{n}{1+\lambda}$$

for every $\lambda > 0$ and $\alpha \in [3-2\sqrt{2}, 3+2\sqrt{2}]$. The desired inequality corresponds to $\lambda = 3$ and $\alpha = 6/7$. $\square$

V3-**2** *Proposed by the editor*

If F_n, L_n and T_n are the $n-$th Fibonacci[3], Lucas[4] and Triangular[5] number respectively, show that

$$2F_n^5 + n^2 T_n F_n - L_{n+1} = \begin{cases} 0 \pmod 5 & , n \neq 0 \pmod 5 \\ 2 \pmod 5 & , n = 5 \pmod{20} \\ 1 \pmod 5 & , n = 10 \pmod{20} \\ 3 \pmod 5 & , n = 15 \pmod{20} \\ 4 \pmod 5 & , n = 0 \pmod{20} \end{cases}, \qquad n \geq 1.$$

Solution 1: Omran Kouba, *Higher Institute for Applied Sciences and Technology, Damascus, Syria*

Using the fact that

$$F_{n+m} = F_m F_{n+1} + F_{m-1} F_n$$

and that $F_{20} = 6765$ and $F_{19} = 4181$ we see that, for every $n \geq 0$ we have $F_{n+20} = F_n \pmod 5$. This means that the sequence $(F_n \pmod 5)_{n\geq 0}$ has 20 as a period.

On the other hand, clearly, for $n \geq 0$, we have

$$T_{n+5} - T_n = \frac{(n+5)(n+6) - n(n+1)}{2} = 5n + 15 = 0 \pmod 5.$$

Thus the sequence $\left(n^2 T_n \pmod 5\right)_{n\geq 0}$ has 5 as a period.

Also, Since $L_4 = 7 = 2 = L_0 \pmod 5$ and $L_5 = 11 = 1 = L_1 \pmod 5$, we can show by an easy induction that $L_{n+4} = L_n \pmod 5$ for every $n \geq 0$. So, the sequence $(L_{n+1} \pmod 5)_{n\geq 0}$ has 4 as a period.

[3] $= \frac{1}{\sqrt{5}}(a^n - b^n)$, where $a = \frac{1+\sqrt{5}}{2}$, $b = \frac{1-\sqrt{5}}{2}$ see `http://en.wikipedia.org/wiki/Fibonacci_number`

[4] $= a^n + b^n$, where $a = \frac{1+\sqrt{5}}{2}$, $b = \frac{1-\sqrt{5}}{2}$, see `http://en.wikipedia.org/wiki/Lucas_number`

[5] $= \frac{n(n+1)}{2}$, see `http://en.wikipedia.org/wiki/Triangular_number`

Let us consider $U_n = 2F_n^5 + n^2T_nF_n - L_{n+1}$. Since 5 is a prime we know that $F_n^5 = F_n \pmod 5$, so

$$U_n = (2 + n^2T_n)F_n - L_{n+1} \pmod 5.$$

The discussion above shows that U_n has 20 as a period. That is $U_{n+20} = U_n$ for every $n \geq 0$. Therefore, to prove the desired conclusion we only need to verify that it holds for $n = 0, 1, \ldots, 19$. But this is a straightforward verification :

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
F_n	0	1	1	2	3	0	3	3	1	4	0	4	4	3	2	0	2	2	4	1
n^2T_n	0	1	2	4	0	0	1	2	4	0	0	1	2	4	0	0	1	2	4	0
L_{n+1}	1	3	4	2	1	3	4	2	1	3	4	2	1	3	4	2	1	3	4	2
U_n	4	0	0	0	0	2	0	0	0	0	1	0	0	0	0	3	0	0	0	0

(All the results are modulo 5.) This concludes the solution of the problem. □

Solution 2: *By the proposer*

We set $a := \dfrac{1+\sqrt{5}}{2}$ and $b := \dfrac{1-\sqrt{5}}{2}$.

Consider the ring $R = \left\{ \dfrac{p+q\sqrt{5}}{r} : p, q \in \mathbb{Z},\ 5 \nmid r \in \mathbb{N}^* \right\}$ and the field $\mathbb{F}_5$. It is straightforward to check that $f : R \to \mathbb{F}_5$ with $\dfrac{p+q\sqrt{5}}{r} \mapsto pr^{-1}$ is a ring homomorphism and that

$$f(a) = f(b) = 2^{-1} = 3.$$

We have $F_0 = 0$ and for $n \geq 1$:

$$\mathbb{Z} \ni F_n - 2n3^n = \frac{a^n - b^n}{a - b} = a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1} - 2n3^n \in R,$$

so, since $f(F_n - 2n3^n) = n3^{n-1} - 2n3^n = 0$ and $\sqrt{5}$ is irrational, we get that

$$F_n = 2n3^n \pmod 5 \qquad \text{for} \qquad n \geq 0. \tag{i}$$

Now it is easy to see, using the fact that a and b satisfy the equation $x^2 - x - 1 = 0$, that

$$L_n = F_{n+1} + F_{n-1} \qquad \text{for} \qquad n \geq 1, \tag{ii}$$

so from (i), this gives that

$$\begin{aligned} L_n &= 2(n+1)3^{n+1} + 2(n-1)3^{n-1} \\ &= 3^{n-1}\left(2(n+1)3^2 + 2n - 2\right) = 3^{n-1}(5n+1) \\ &= 3^{n-1} \pmod 5 \qquad \text{for} \qquad n \geq 1. \end{aligned} \tag{iii}$$

Now from Fermat's theorem and from (ii) and (iii) we get that

$$\begin{aligned}
2F_n^5 + n^2T_nF_n - L_{n+1} &= 2F_n + n^2T_nF_n - 3^n \\
&= 4n3^n + n^2\frac{n(n+1)}{2}2n3^n - 3^n \\
&= 4n3^n + (n^5+n^4)3^n - 3^n \\
&= 3^n\left(n^4+5n-1\right) \\
&= 3^n\left(n^4-1\right) \\
&= \begin{cases} 0 \pmod 5 & , n \neq 0 \pmod 5 \\ 3^{n+2} \pmod 5 & , n = 0 \pmod 5 \end{cases}
\end{aligned}$$

But, since $3^4 = 1 \pmod 5$,

- For $n = 5 \pmod{20}$: $3^{n+2} = 2 \pmod 5$
- For $n = 10 \pmod{20}$: $3^{n+2} = 1 \pmod 5$
- For $n = 15 \pmod{20}$: $3^{n+2} = 3 \pmod 5$
- For $n = 0 \pmod{20}$: $3^{n+2} = 4 \pmod 5$

which completes the proof. □

V3-3 Evaluate $\displaystyle\sum_{n\geq 1}(-1)^n\frac{\prod_{j=1}^{n}\left(\frac{3}{2}-j\right)}{(2n+1)n!}$ if it converges.

Solution 1: Omran Kouba, *Higher Institute for Applied Sciences and Technology, Damascus, Syria*

Recall that, for $x \in [-1,1]$

$$\sum_{n=1}^{\infty}\frac{\prod_{j=1}^{n}\left(\frac{3}{2}-j\right)}{n!}(-x)^n = \sqrt{1-x}-1$$

It follows that

$$\sum_{n=1}^{\infty}(-1)^n\frac{\prod_{j=1}^{n}\left(\frac{3}{2}-j\right)}{n!}x^{2n} = \sqrt{1-x^2}-1.$$

Integrating on $[0,1]$ we obtain

$$\sum_{n=1}^{\infty}(-1)^n\frac{\prod_{j=1}^{n}\left(\frac{3}{2}-j\right)}{n!(2n+1)} = \int_0^1(\sqrt{1-x^2}-1)\,dx = \frac{\pi}{4}-1$$

which is the desired conclusion. □

Also solved by *Moubinool Omarjee, Lycée Henri IV, Paris, France*

V3-4 Let a_n be the sequence defined by $a_{n+1} = a_n + a_n^{-k}, \quad a_1 > 0,\ k \in \mathbb{R}$.

1. Show that, for $k > -1$:

$$a_n = (k+1)^{\frac{1}{k+1}} n^{\frac{1}{k+1}} \left(1 + \frac{k}{2(k+1)^2} \frac{\ln n}{n} + \mathcal{O}\left(n^{-1}\right)\right)$$

2. (*) Can we make a two terms estimate, as in 1., when $k < -1$?

Solution 1: Omran Kouba, *Higher Institute for Applied Sciences and Technology, Damascus, Syria*

1. In this part we assume that $k > -1$. Clearly the sequence $(a_n)_{n\geq 1}$ is strictly increasing, and it cannot converge to a finite limit ℓ, because if it does we would have $0 = 1/\ell^k$ which is absurd. Thus, we must have $\lim\limits_{n\to\infty} a_n = +\infty$, and consequently

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(1 + \frac{1}{a_n^{k+1}}\right) = 1$$

Now, according to the Mean Value Theorem, for every $n \geq 1$ there exists some $\xi_n \in (a_n, a_{n+1})$ such that

$$a_{n+1}^{k+1} - a_n^{k+1} = (k+1)(a_{n+1} - a_n)\xi_n^k = (k+1)\left(\frac{\xi_n}{a_n}\right)^k$$

It follows that

$$\lim_{n\to\infty} \left(a_{n+1}^{k+1} - a_n^{k+1}\right) = k+1,$$

and Stolz-Cezàro's Theorem implies that

$$\lim_{n\to\infty} \frac{b_n}{n} = k+1.$$

where we wrote b_n to denote a_n^{k+1} for simplicity, and we have shown that $b_n \sim (k+1)n$.

Now,

$$\begin{aligned} b_{n+1} = \left(a_n + \frac{1}{a_n^k}\right)^{k+1} &= b_n \left(1 + \frac{1}{b_n}\right)^{k+1} \\ &= b_n \left(1 + \frac{k+1}{b_n} + \frac{(k+1)k}{2b_n^2} + O\left(\frac{1}{b_n^3}\right)\right) \\ &= b_n + k + 1 + \frac{(k+1)k}{2b_n} + O\left(\frac{1}{b_n^2}\right) \\ &= b_n + k + 1 + \frac{(k+1)k}{2b_n} + O\left(\frac{1}{n^2}\right) \qquad (*) \end{aligned}$$

It follows that,

$$\frac{b_{n+1} - b_n - (k+1)}{\ln(n+1) - \ln(n)} = \frac{1}{n\,\ln(1+1/n)} \left(\frac{k}{2}\frac{(k+1)n}{b_n} + O\left(\frac{1}{n}\right)\right).$$

Consequently
$$\lim_{n\to\infty}\frac{b_{n+1}-b_n-(k+1)}{\ln(n+1)-\ln n}=\frac{k}{2}.$$
Again, Stolz-Cezàro's Theorem implies that
$$\lim_{n\to\infty}\frac{b_n-(k+1)n}{\ln n}=\frac{k}{2}.$$
It follows that $b_n=(k+1)n+\frac{k}{2}\ln n+o(\ln n)$, so
$$\frac{1}{b_n}=\frac{1}{(k+1)n}+O\left(\frac{\ln n}{n^2}\right)=\frac{\ln(n+1)-\ln n}{k+1}+O\left(\frac{\ln n}{n^2}\right)$$
Replacing back in (*) proves that
$$b_{n+1}=b_n+k+1+\frac{k}{2}(\ln(n+1)-\ln n)+O\left(\frac{\ln n}{n^2}\right)$$
So, the series
$$\sum_{n=1}^{\infty}\left(b_{n+1}-b_n-(k+1)-\frac{k}{2}(\ln(n+1)-\ln n)\right)$$
is convergent, and there is a real number ℓ such that
$$\ell=\lim_{n\to\infty}\left(b_n-(k+1)n-\frac{k}{2}\ln n\right).$$
In particular,
$$\begin{aligned}a_n^{k+1}=b_n&=(k+1)n+\frac{k}{2}\ln n+\ell+o(1)\\&=(k+1)n\left(1+\frac{k}{2(k+1)}\frac{\ln n}{n}+\mathcal{O}(n^{-1})\right)\end{aligned}$$
and finally
$$a_n=\sqrt[k+1]{(k+1)n}\left(1+\frac{k}{2(k+1)^2}\frac{\ln n}{n}+\mathcal{O}(n^{-1})\right),$$
which is the desired conclusion in this case.

2. (Partial solution.) Here we suppose that $k=-p$ for some $p>1$. In this case, it is clear that the sequence $(a_n)_{n\geq1}$ is strictly increasing and $\lim\limits_{n\to\infty}a_n=+\infty$. In particular, there exists n_0 such that $a_n>1$ for every $n\geq n_0$.

   Now, let $z_n=a_n^{1/p^n}$, for $n\geq1$. From $a_{n+1}=a_n+a_n^p$ we conclude that, for $n\geq n_0$, we have $a_n^p\leq a_{n+1}\leq 2a_n^p$ so
$$\forall\,n\geq n_0,\qquad 1<\frac{z_{n+1}}{z_n}<2^{1/p^n}$$
   From the convergence of $\sum\limits_{n\geq1}p^{-n}$, we conclude that the infinite product $\prod\limits_{n\geq1}\frac{z_{n+1}}{z_n}$ is convergent, and this proves the existence of a constant $\xi\geq z_{n_0}>1$, (that depends on a_1,) such that $\lim\limits_{n\to\infty}z_n=\xi$. *i.e.*
$$\lim_{n\to\infty}a_n^{1/p^n}=\xi.$$
   This is as much as I could get. □

Solution 2: *Moubinool Omarjee, Lycée Henri IV, Paris, France*

1. All terms of the sequence are strictly positive and from the recurrence we get that the sequence is strictly increasing. If a_n converged to ℓ, then from the recurrence we would get that $\ell = \ell + \ell^{-k}$ which is absurd, so $a_n \to +\infty$.
   Now consider $a_{n+1}^{k+1} - a_n^{k+1}$. We have

$$\begin{aligned} a_{n+1}^{k+1} - a_n^{k+1} &= a_n^{k+1}\left(\left(1 + a_n^{-k-1}\right)^{k+1} - 1\right) \\ &= a_n^{k+1}\left(1 + \frac{k+1}{a_n^{k+1}} + \frac{(k+1)k}{2a_n^{2k+2}} + O\left(a_n^{-3k-3}\right) - 1\right) \\ &= k+1+\frac{(k+1)k}{2a_n^{k+1}} + O\left(a_n^{-2k-2}\right) \end{aligned} \tag{1}$$

   so $a_{n+1}^{k+1} - a_n^{k+1} \to k+1$, since $a_n^{-k-1} \to 0$. Now from Cesàro Stolz theorem we have

$$\frac{1}{n-1}\sum_{j=1}^{n-1}(a_{j+1}^{k+1} - a_j^{k+1}) \to k+1,$$

   and since $\sum_{j=1}^{n-1}(a_{j+1}^{k+1} - a_j^{k+1}) = a_n^{k+1} - a_1^{k+1}$ we deduce that $\frac{a_n^{k+1}}{n} \sim k+1$, or $a_n^{k+1} = (k+1)n + o(n)$, so $a_n = (k+1)^{\frac{1}{k+1}} n^{\frac{1}{k+1}}(1+o(1))$ and hence $a_n \sim (k+1)^{\frac{1}{k+1}} n^{\frac{1}{k+1}}$.
   From (1) we get

$$a_{n+1}^{k+1} - a_n^{k+1} - (k+1) \sim \frac{(k+1)k}{2a_n^{k+1}} \sim \frac{(k+1)k}{2(k+1)n} = \frac{k}{2n}$$

   and by the equivalence of the partial sums of positive divergent series we have

$$\sum_{j=1}^{n-1}\left(a_{j+1}^{k+1} - a_j^{k+1} - (k+1)\right) \sim \sum_{j=1}^{n}\frac{k}{2j} \sim \frac{k}{2}\ln n,$$

   so

$$\begin{aligned} a_n^{k+1} - a_1^{k+1} - (k+1)n &\sim \frac{k}{2}\ln n \Rightarrow \\ a_n^{k+1} &= (k+1)n + \frac{k}{2}\ln n + o\left(\ln n\right) \Rightarrow \\ a_n &= ((k+1)n)^{\frac{1}{k+1}}\left(1 + \frac{k\ln n}{2(k+1)n} + o\left(n^{-1}\ln n\right)\right)^{\frac{1}{k+1}} \Rightarrow \\ a_n &= ((k+1)n)^{\frac{1}{k+1}}\left(1 + \frac{k\ln n}{2(k+1)^2 n} + o\left(n^{-1}\ln n\right)\right) \end{aligned}$$

   Again from (1) we have

$$\sum_{j=1}^{n-1}\left(a_{j+1}^{k+1} - a_j^{k+1} - (k+1)\right) = \sum_{j=1}^{n-1}\frac{(k+1)k}{2a_j^{k+1}} + O\left(\sum_{j=1}^{n-1} a_j^{-2k-2}\right)$$

and since $a_j^{k+1} \sim (k+1)j$,

$$\sum_{j=1}^{n}\left(a_{j+1}^{k+1}-a_j^{k+1}-(k+1)\right)=\sum_{j=1}^{n-1}\frac{(k+1)k}{2a_j^{k+1}}+O\left(1\right).$$

Furthermore, from $a_n^{k+1}=(k+1)n+\dfrac{k}{2}\ln n+o\left(\ln n\right)$:

$$\begin{aligned}
\sum_{j=1}^{n-1}\frac{(k+1)k}{2a_j^{k+1}} &= \frac{(k+1)k}{2}\sum_{j=1}^{n-1}\frac{1}{(k+1)j+\frac{k}{2}\ln j+o\left(\ln j\right)}\\
&= \frac{k}{2}\sum_{j=1}^{n-1}\frac{1}{j}\cdot\frac{1}{1+\frac{k}{2(k+1)}\cdot\frac{\ln j}{j}+o\left(\frac{\ln j}{j}\right)}\\
&= \frac{k}{2}\sum_{j=1}^{n-1}\left(1-\frac{k}{2(k+1)}\cdot\frac{\ln j}{j}+o\left(\frac{\ln j}{j}\right)\right)\\
&= \frac{k}{2}\sum_{j=1}^{n-1}\frac{1}{j}-\frac{k^2}{2(k+1)}\sum_{j=1}^{n-1}\frac{\ln j}{j^2}+o\left(\sum_{j=1}^{n-1}\frac{\ln j}{j^2}\right)\\
&= \frac{k}{2}\sum_{j=1}^{n-1}\frac{1}{j}+O(1)=\frac{k}{2}\ln n+O(1)
\end{aligned}$$

Now

$$\begin{aligned}
\sum_{j=1}^{n-1}\left(a_{j+1}^{k+1}-a_j^{k+1}-(k+1)\right)=\sum_{j=1}^{n-1}\frac{(k+1)k}{2a_j^{k+1}}&=\frac{k}{2}\ln n+O(1)\Rightarrow\\
a_n^{k+1}-(k+1)n&=\frac{k}{2}\ln n+O(1)\Rightarrow\\
a_n&=((k+1)n)^{\frac{1}{k+1}}\left(1+\frac{k}{2(k+1)}\frac{\ln n}{n}+O\left(n^{-1}\right)\right)^{\frac{1}{k+1}}\Rightarrow\\
a_n&=((k+1)n)^{\frac{1}{k+1}}\left(1+\frac{k}{2(k+1)^2}\frac{\ln n}{n}\right)+O\left(n^{-1}\right)
\end{aligned}$$

□

Remark by the editor: The first part of the problem has also been discussed on the Greek mathematical forum `www.mathematica.gr` (`http://www.mathematica.gr/forum/viewtopic.php?f=9&t=27892`).

Several examples on asymptotics of recurrent sequences can be found in [1] and also in [2] and [3].

V3-5 Show that for $n \in \mathbb{N}^*$:

$$\sum_{r\geq 1} \frac{(-1)^r}{r\binom{n+r}{n}} = 2^n \ln 2 + \sum_{k=0}^{n-1}(-1)^{n-k}\binom{n}{k}\frac{2^n - 2^k}{n-k}.$$

Solution 1: Omran Kouba, *Higher Institute for Applied Sciences and Technology, Damascus, Syria*

Note that

$$\frac{1}{r\binom{n+r}{n}} = \frac{n!(r-1)!}{(n+r)!} = \int_0^1 (1-t)^n t^{r-1}\,dt$$

So

$$\begin{aligned}\sum_{r=1}^{m} \frac{(-1)^{r-1}}{r\binom{n+r}{n}} &= \int_0^1 (1-t)^n \left(\sum_{r=1}^{m}(-t)^{r-1}\right)\,dt \\ &= \int_0^1 (1-t)^n \frac{1-(-t)^m}{1+t}\,dt = \int_0^1 (1-t)^n \frac{1}{1+t}\,dt - R_m\end{aligned}$$

with

$$|R_m| = \int_0^1 (1-t)^n \frac{t^m}{1+t}\,dt \leq \int_0^1 t^m\,dt = \frac{1}{m+1}.$$

So, letting m tend to infinity we find that

$$\begin{aligned}\sum_{r=1}^{\infty} \frac{(-1)^r}{r\binom{n+r}{n}} &= \int_0^1 (1-t)^n \frac{1}{1+t}\,dt = \int_1^2 (2-t)^n \frac{1}{t}\,dt \\ &= 2^n \int_1^2 \frac{1}{t}\,dt + \sum_{k=0}^{n-1}\binom{n}{k}2^k(-1)^{n-k}\int_1^2 t^{n-k-1}dt \\ &= 2^n \ln 2 + \sum_{k=0}^{n-1}\binom{n}{k}2^k(-1)^{n-k}\frac{2^{n-k}-1}{n-k} \\ &= 2^n \ln 2 + \sum_{k=0}^{n-1}(-1)^{n-k}\binom{n}{k}\frac{2^n-2^k}{n-k}.\end{aligned}$$

which is the desired result. □

Solution 2: *By the editor*

We have

$$
\begin{aligned}
\sum_{r\geq 1}\frac{(-1)^r}{r\binom{n+r}{n}} &= \sum_{r\geq 1}\frac{(-1)^r}{r}(n+r+1)\int_0^1 t^n(1-t)^r\,dt\\
&\overset{6}{=} -\int_0^1 t^n\left((n+1)\sum_{r\geq 1}\frac{(-1)^{r-1}}{r}(1-t)^r-\sum_{r\geq 1}(t-1)^r\right)\,dt\\
&= -\int_0^1 t^n\left((n+1)\ln(2-t)+\frac{1-t}{2-t}\right)\,dt\\
&\overset{1-t=x}{=} -\int_0^1\frac{x(1-x)^n}{1+x}\,dx-(n+1)\int_0^1(1-x)^n\ln(1+x)\,dx\\
&= -\int_0^1\frac{(1+x-1)(1-x)^n}{1+x}\,dx-(n+1)\int_0^1(1-x)^n\ln(1+x)\,dx\\
&= -\int_0^1(1-x)^n\,dx+\int_0^1\frac{(1-x)^n}{1+x}\,dx-(n+1)\int_0^1\left(-\frac{(1-x)^{n+1}}{n+1}\right)'\ln(1+x)\,dx\\
&= -\frac{1}{n+1}+\int_0^1\frac{(1-x)^n}{1+x}\,dx-\int_0^1\frac{(1-x)^{n+1}}{1+x}\,dx\\
&= -\frac{1}{n+1}+\int_0^1\frac{x(1-x)^n}{1+x}\,dx
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{n+1}+\int_0^1\frac{(1+x-1)(1-x)^n}{1+x}\,dx\\
&= -\frac{1}{n+1}+\int_0^1(1-x)^n\,dx-\int_0^1\frac{(1-x)^n}{1+x}\,dx\\
&= -\int_0^1\frac{(1-x)^n}{1+x}\,dx\\
&\overset{1+x=y}{=} -2^{n-1}\int_1^2\frac{\left(1-\frac{y}{2}\right)^n}{\frac{y}{2}}\,dt\\
&\overset{y=2t}{=} 2^n\int_{1/2}^1\frac{(1-t)^n}{t}\,dt\\
&= 2^n\sum_{k=0}^{n}(-1)^k\binom{n}{k}\int_{1/2}^1 t^{k-1}\,dt\\
&= 2^n\ln 2+\sum_{k=1}^{n}(-1)^k\binom{n}{k}\frac{2^n-2^{n-k}}{k}\\
&\overset{n-k+1\mapsto k}{=} 2^n\ln 2+\sum_{k=1}^{n}(-1)^{n+1-k}\binom{n}{n+1-k}\frac{2^n-2^{k-1}}{n+1-k}\\
&= 2^n\ln 2+\sum_{k=0}^{n-1}(-1)^{n-k}\binom{n}{k}\frac{2^n-2^k}{n-k}
\end{aligned}
$$

[6]The change of the summation and itegration sign is justified by the uniform convergence of the corresponding series on $[0,1]$ for $n\in\mathbb{N}$.

$\square$

Remark by the editor: A different approach is presented in [4] p.57.

V3-6 Let $F(x) := \int_0^{+\infty} \frac{1}{e^t + xt}\, dt$ for the values of $x \in \mathbb{R}$ for which it can be defined.

1. Find the MacLaurin expansion of $F(x)$ at 0, if it has one, and determine it's radius of convergence.
2. Show that $\lim_{x \to -e^+} (x+e)^{1/2} \int_0^{+\infty} \frac{1}{e^t + xt}\, dt = \pi\sqrt{\frac{2}{e}}$.
3. (*) Examine whether there exists a real number $a < 0$ such that

$$\lim_{x \to -e^+} (x+e)^a \left((x+e)^{1/2} \int_0^{+\infty} \frac{1}{e^t + xt}\, dt - \pi\sqrt{\frac{2}{e}} \right) \in \mathbb{R}^*$$

and, for this real number a, in the case it exists, compute the limit.

Solution 1: Omran Kouba, *Higher Institute for Applied Sciences and Technology, Damascus, Syria*

1. Studying the variations of $\varphi : [0, +\infty) \longrightarrow \mathbb{R}, \varphi(t) = te^{-t}$, we see easily that φ takes its values in the interval $[0, 1/e]$. Readily, this allows us to see that $F(x) = \int_0^{+\infty} \frac{e^{-t}}{1 + x\varphi(t)}\, dt$ is defined if and only if $x > -e$.

Now, consider $x \in (-e, e)$. Since

$$\left| \frac{e^{-t}}{1 + x\varphi(t)} - \sum_{n=0}^{m-1} (-x)^n \varphi^n(t) e^{-t} \right| \leq \frac{|x|^m}{1 - |x|/e} \varphi^m(t) e^{-t},$$

and

$$\int_0^{\infty} \varphi^n(t) e^{-t}\, dt = \int_0^{\infty} t^n e^{-(n+1)t}\, dt = \frac{1}{(n+1)^{n+1}} \int_0^{\infty} s^n e^{-s}\, ds = \frac{n!}{(n+1)^{n+1}}$$

we conclude that, for $m > 0$, we have

$$\left| F(x) - \sum_{n=0}^{m-1} \frac{n!(-1)^n}{(n+1)^{n+1}} x^n \right| \leq \frac{|x|^m}{1 - |x|/e} \cdot \frac{m!}{(m+1)^{m+1}}, \tag{1}$$

Now, using Stirling's formula, we see that $\frac{m!}{(m+1)^{m+1}} \sim \frac{\sqrt{2\pi}}{e} \frac{e^{-m}}{\sqrt{m}}$. So, recalling that $|x| < e$, and letting m tend to $+\infty$ in (1) we conclude that

$$\forall\, x \in (-e, e), \qquad F(x) = \sum_{n=0}^{\infty} \frac{n!(-1)^n}{(n+1)^{n+1}} x^n$$

which is the MacLaurin expansion of F at 0 and its radius of onvergence is e, this answers 1.

2. and 3. Note that for $t \in (-1,1)$, we have $F(-et) = \sum_{n=0}^{\infty} a_n t^n$ where

$$a_n = \frac{n!e^n}{(n+1)^{n+1}} = \frac{\sqrt{2\pi}}{e\sqrt{n+1}} \cdot \frac{(n+1)!}{\sqrt{2\pi(n+1)}(n+1)^{n+1}e^{-(n+1)}} \tag{2}$$

Consider also

$$b_n = \frac{\pi\sqrt{2}}{e} \cdot \frac{(2n)!}{2^{2n}(n!)^2} \tag{3}$$

Starting from the well-known expansion $n! = \sqrt{2\pi n}\, n^n\, e^{-n}\left(1 + O\left(\frac{1}{n}\right)\right)$, we see that

$$\begin{aligned} a_n &= \frac{\sqrt{2\pi}}{e\sqrt{n+1}}\left(1 + O\left(\frac{1}{n}\right)\right) \\ &= \frac{\sqrt{2\pi}}{e\sqrt{n}}\left(1 + \frac{1}{n}\right)^{-1/2}\left(1 + O\left(\frac{1}{n}\right)\right) \\ &= \frac{\sqrt{2\pi}}{e\sqrt{n}}\left(1 + O\left(\frac{1}{n}\right)\right) \end{aligned} \tag{4}$$

and similarly,

$$b_n = \frac{\sqrt{2\pi}}{e\sqrt{n}}\left(1 + O\left(\frac{1}{n}\right)\right) \tag{5}$$

So, $a_n - b_n = O(n^{-3/2})$ and the series $\sum_{n=0}^{\infty}(a_n - b_n)$ is absolutely convergent. Let

$$\ell = \sum_{n=0}^{\infty}(a_n - b_n). \tag{6}$$

The function G defined by $G(t) = \sum_{n=0}^{\infty}(a_n - b_n)t^n$ for $t \in [-1,1]$ is continuous on this interval and $\lim_{t\to 1^-} G(t) = G(1) = \ell$.

But, for $t \in (-1,1)$ we have

$$G(t) = \sum_{n=0}^{\infty} a_n t^n - \frac{\pi\sqrt{2}}{e}\sum_{n=0}^{\infty}\frac{1}{2^{2n}}\binom{2n}{n}t^n = F(-et) - \frac{\pi\sqrt{2}}{e}\frac{1}{\sqrt{1-t}}$$

Or equivalently,

$$\forall\, x \in (-e,e), \quad F(x) = \pi\sqrt{\frac{2}{e}} \cdot \frac{1}{\sqrt{e+x}} + G\left(-\frac{x}{e}\right)$$

Now, the fact that $\lim_{x\to(-e)^+} G(-x/e) = \ell$, proves that

$$F(x) = \pi\sqrt{\frac{2}{e}} \cdot \frac{1}{\sqrt{e+x}} + \ell + o(x+e), \qquad \text{as } x \to (-e)^+$$

This implies .2, and also proves partially .3 with $a = -1/2$. The desired limit ℓ is given by (6). Now, noting that $b_n = c_{n+1} - c_n$ where $c_n = \frac{\pi\sqrt{2}}{e} \cdot n2^{1-2n}\binom{2n}{n}$ we see that $\ell = \lim_{n\to\infty} \ell_n$ where

$$\begin{aligned}
\ell_n &= \sum_{m=0}^{n-1} a_m - \sum_{m=0}^{n-1} b_m = \sum_{m=0}^{n-1} a_m - c_n \\
&= \frac{\sqrt{2\pi}}{e}\left(\sum_{m=1}^{n} \frac{1}{\sqrt{m}} \cdot \frac{m!}{\sqrt{2\pi m}m^m e^{-m}} - 2\sqrt{n}\right) \\
&= \frac{\sqrt{2\pi}}{e}\left(\sum_{m=1}^{n} \frac{1}{\sqrt{m}} \cdot \left(\frac{m!}{\sqrt{2\pi m}m^m e^{-m}} - 1\right) + \sum_{m=1}^{n} \frac{1}{\sqrt{m}} - 2\sqrt{n}\right)
\end{aligned}$$

where we used the fact that $c_n = \frac{2\sqrt{2\pi}}{e}\sqrt{n}\left(1 + O(\frac{1}{n})\right)$. It follows that

$$\ell = \frac{\sqrt{2\pi}}{e}\left(\sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \cdot \left(\frac{m!}{\sqrt{2\pi m}m^m e^{-m}} - 1\right) + \zeta\left(\frac{1}{2}\right)\right)$$

where we used the well known fact $\zeta(\frac{1}{2}) = \lim_{n\to\infty}\left(\sum_{k=1}^{n} \frac{1}{\sqrt{k}} - 2\sqrt{n}\right)$. We can accelerate the convergence of the series defining ℓ using the asymptotic expansion of $\frac{m!}{\sqrt{2\pi m}m^m e^{-m}} - 1$. This gives the value $\ell \approx -1.1440964591103915$. This is a partial answer since it "computes" only 16 digits of the desired limit!. But the question remains : Can one express ℓ in termes of some other well-known constants ? ☐

Remark by the editor: The first part of this problem was Spring 2013, problem 7, of the Problem of the Week page of Mathematics Department of Purdue University. A solution to this part by the editor can be found in (`http://www.asymmetry.gr/index.php?option=com_content&view=article&id=8:2013-09-23-17-29-07gb&catid=8:maths&lang=en&Itemid=491`).

The second part of the problem has also been discussed on the Art Of Problem Solving forum (`http://www.artofproblemsolving.com/Forum/viewtopic.php?f=67&t=521949`).

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